Uniform generator of the Brauer group of affine diagonal quadrics

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Outline

Our result

Setting

- k: a field of characteristic zero.
- Assume $\exists k'/k$ s.t $[k' : k] = 2$.
- $U_{b,c,d} = \operatorname{Spec} k[x,y,z]/(x^2 + by^2 + cz^2 + d) \xrightarrow{\pi} \operatorname{Spec} k$.
- $\pi^* \colon \operatorname{Br}(k) \to \operatorname{Br}(U_{b,c,d})$.
- $\operatorname{Br}(U_{b,c,d}) / \operatorname{Br}(k) := \operatorname{Coker} \pi^*$.

Theorem (U-.)

There is no uniform generator for the Brauer group $\operatorname{Br}(U_{b,c,d}) / \operatorname{Br}(k)$.

What is uniform generator? – Naive Example

Question

Can we solve the following equation

$$\boldsymbol{Ax^2 + Bx + C = 0} \ ?$$

Answer

Of course YES! Moreover, we can find the following uniform algebraic solution:

$$\boldsymbol{x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}}.$$

Uniform generator

generator which is "algebraically parametrized".

Brauer group of affine diagonal quadrics

Proposition (Structure of $\mathrm{Br}(U_{b,c,d})$)

$$\mathrm{Br}(U_{b,c,d})/\,\mathrm{Br}(k) \cong \mathbb{Z}/2\mathbb{Z} \quad \text{or} \quad 0.$$

Definition (Domain of parameters)

$$\mathcal{P}_k := \{(b,c,d) \mid \mathrm{Br}(U_{b,c,d})/\,\mathrm{Br}(k) \cong \mathbb{Z}/2\mathbb{Z}.\}$$

Remark

By the assumption $\exists k'$ s.t. $[k' : k] = 2$, we assure that $\mathcal{P}_k$ is Zariski dense in $\mathbb{A}^3_k$.

Formulation of uniform generator

Setting

- $\mathcal{O}_F = k[B, C, D]$.
- $\mathbb{A}^3_k = \operatorname{Spec} \mathcal{O}_F$.
  (3-dimensional parameter space)
- $F = \operatorname{Frac} \mathcal{O}_F = k(B, C, D)$.
- $\mathcal{U} = \{x^2 + By^2 + Cz^2 + D = 0\}$ over $\mathbb{A}^3_k$,
  i.e. a 3-parameter family of affine diagonal quadrics over k.
- $U = \{x^2 + By^2 + Cz^2 + D = 0\}$ over F.
- For $P = (b, c, d) \in k^* \times k^* \times k^*$,
  $U_P = \{x^2 + by^2 + cz^2 + d = 0\}$ over k.

Lemma & Definition(Specialization map)

- $\forall e \in \mathrm{Br}(U)$, $\exists$ (Zariski) dense open $W \subset \mathbb{A}^3_k$, $\exists \widetilde{e} \in \mathrm{Br}(\mathcal{U}_W)$ s.t. $\widetilde{e}|_U = e$.
- $\forall P \in W(k)$, we define $\mathrm{sp}(e; P) \in \mathrm{Br}(U_P)$ by $P^*\widetilde{e}$. We call $\mathrm{sp}(e; P)$ the specialization of e at P.

Definition (Uniform generator)

$e \in \mathrm{Br}(U)$ is a uniform generator
$\overset{\text{def}}{\Leftrightarrow}$ $\exists$ dense open $W \subset \mathbb{A}^3_k$ s.t. $\forall P \in W(k) \cap \mathcal{P}_k$,
$\mathrm{sp}(e; P) \in \mathrm{Br}(U_P)/\,\mathrm{Br}(k)$ is its generator.

Main Theorem

Recall:

- $\mathcal{O}_F = k[B, C, D], \quad F = k(B, C, D)$.
- $U = \{x^2 + By^2 + Cz^2 + D = 0\}$ over F.

Theorem (U-.)

For this 3-parameter family U, there is no uniform generator.

Remark

The following 2-parameter family

$$V = \{x^2 - y^2 - Cz^2 + D = 0\} \text{ over } k(C, D)$$

has a uniform generator.

Thank you for your attention!