アフィン対角的２次曲面の Brauer 群について (On the Brauer group of affine diagonal quadrics)

Tetsuya Uematsu

National Institute of Technology, Toyota College

March 04, 2015.
11th MCYR (Hokkaido University)

Outline

Our Result

Theorem (U-.)

There is no uniform generator for the Brauer group of affine diagonal quadrics.

What is ...

- Brauer group ?
- Uniform generator ?
- Affine diagonal quadric ?

What is Brauer group?

Brauer group

- X: an algebraic variety
  (i.e. a topological space defined by common zeros of algebraic equations)
  $\rightsquigarrow$ $\mathbf{Br}(X)$: the Brauer group of X.
- $\mathbf{Br}(X)$ has many applications to geometry and arithmetic.
  - rationality problem, Hasse principle, Chow groups, etc...

Basic questions for $\mathbf{Br}(X)$

- How about its group structure? (Note: abelian by definition)
- How can we express its generator(s) in a useful fashion?

What is affine diagonal quadric?

Affine diagonal quadric

X: an affine diagonal quadric,
$\overset{\text{def}}{\Leftrightarrow}$ an affine variety in $\mathbb{A}^3$ defined by the following equation

$$x^2 + By^2 + Cz^2 + D = 0,$$

where $B, C, D \neq 0$.

What is uniform generator? – Naive Example

Question

Can we solve the following equation

$$Ax^2 + Bx + C = 0 \ ?$$

Answer

Of course YES!

Moreover, we can find the following uniform algebraic solution:

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

(if we assume the characteristic of a field is not equal to 2.)

Uniform generator

generator which is "algebraically parametrized".

Question

Question

Can we find a uniform generator of $\mathrm{Br}(\boldsymbol{X})$?

Answer

NO!

Brauer group of fields

Definition

- k: a field
  $\rightsquigarrow$ the Brauer group $\mathrm{Br}(k)$ of k is defined as the classifying space of Morita equivalent classes of isomorphism classes of central simple algebras over k.
  -We can take the class of a central skew field over k as each representative.
- $\mathrm{Br}(k)$ can be defined as follows:

$$\mathrm{Br}(k) := H^2(G_k, \overline{k}^*),$$

  where
  - $\overline{k}$ = a separable closure of k,
  - $G_k = \mathrm{Gal}(\overline{k}/k)$.
- $\mathrm{Br}(k)$ has many applications to algebra and number theory.

Symbol

- k: a field, n: a positive integer with $(n, \mathrm{ch}\, k) = 1$
- Assume k contains a primitive n-th root ζ_n of unity
- $\mu_n := \langle \zeta_n \rangle$ ($\cong \mathbb{Z}/n\mathbb{Z}$ as G_k-modules)

Definition (Norm residue map)

The n-th norm residue map

$$\{\cdot, \cdot\}_n : k^* \otimes_{\mathbb{Z}} k^* \to {}_n\mathrm{Br}(k)$$

is defined to be the following composite:

$$\begin{aligned} & k^* \otimes_{\mathbb{Z}} k^* \\ & \cong H^1(G_k, \mu_n) \otimes_{\mathbb{Z}} H^1(G_k, \mu_n) && \text{(Kummer seq.)} \\ & \xrightarrow{\cup} H^2(G_k, \mu_n \otimes \mu_n) && \text{(cup product)} \\ & \cong H^2(G_k, \mu_n) && (\mu_n \cong \mathbb{Z}/n\mathbb{Z}) \\ & \cong {}_n\mathrm{Br}(k) && \text{(Kummer seq.)} \end{aligned}$$

Examples

Example

- $\mathbf{Br}(\mathbb{C}) = 0.$
- $\mathbf{Br}(\mathbb{F}) = 0, \quad \mathbf{Br}(\mathbb{C}(t)) = 0.$
- $\mathbf{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}.$

  The only non-trivial element is the class of $\mathbb{H}$:

  $$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\,i \oplus \mathbb{R}\,j \oplus \mathbb{R}\,k$$

  with $i^2 = j^2 = k^2 = ijk = -1.$

  We also have

  $$[\mathbb{H}] = \{-1, -1\}_2$$

- $\mathbf{Br}(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}.$

Brauer group of varieties

Definition (Brauer group of varieties)

For a variety $\pi\colon X \to \operatorname{Spec} k$, define:

- the *Brauer group* of X:

$$\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m).$$

- We have $\pi^*\colon \mathrm{Br}(k) \to \mathrm{Br}(X)$.
- Put $\mathrm{Br}(X)/\mathrm{Br}(k) = \mathrm{Br}(X)/\pi^* \mathrm{Br}(k)$.

$\mathrm{Br}(X)$ v.s. $\mathrm{Br}(k(X))$

- $k(X)$: the function field of X.
- We have $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$.
- If X is smooth, $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k(X))$
  $\rightsquigarrow$ elements of $\mathrm{Br}(X)$ may be expressed by symbols in $\mathrm{Br}(k(X))$.

Brauer group of affine diagonal quadrics

Affine diagonal quadric

Let k be a field of characteristic zero.
$U_{b,c,d}$: affine diagonal quadrics in $\mathbb{A}^3$ defined by

$$x^2 + by^2 + cz^2 + d = 0,$$

where $b, c, d \in k^*$.
(Note that U is (geometrically) rational.)

Proposition (Structure of $\mathrm{Br}(U_{b,c,d})$)

$$\mathrm{Br}(U_{b,c,d}) / \mathrm{Br}(k) \cong \mathbb{Z}/2\mathbb{Z} \text{ or } 0.$$

Definition (Domain of parameters)

$$\mathcal{P}_k := \{(b, c, d) \mid \mathrm{Br}(U_{b,c,d}) / \mathrm{Br}(k) \cong \mathbb{Z}/2\mathbb{Z}.\}$$

Formulation of uniform generator

Setting

- $\mathcal{O}_F = k[B, C, D]$.
- $\mathbb{A}^3_k = \operatorname{Spec} \mathcal{O}_F$.
  (3-dimensional parameter space)
- $F = \operatorname{Frac} \mathcal{O}_F = k(B, C, D)$.
- $\mathcal{U} = \{x^2 + By^2 + Cz^2 + D = 0\}$ over $\mathbb{A}^3_k$,
  i.e. a 3-parameter family of affine diagonal quadrics over k.
- $U = \{x^2 + By^2 + Cz^2 + D = 0\}$ over F.
- For $P = (b, c, d) \in k^* \times k^* \times k^*$,
  $U_P = \{x^2 + by^2 + cz^2 + d = 0\}$ over k.

Formulation of uniform generator

Definition (Specialization map)

- $\forall e \in \mathrm{Br}(U)$(=the Brauer group of the family),
- $\exists$ (Zariski) dense open $W \subset \mathbb{A}_k^3$ s.t.
- $\forall P \in W(k)$, we can define $\mathrm{sp}(e; P) \in \mathrm{Br}(U_P)$.
  We call $\mathrm{sp}(e; P)$ the specialization of e at P.

Definition (Uniform generator)

$e \in \mathrm{Br}(U)$ is a uniform generator
$\overset{\text{def}}{\Leftrightarrow}$ $\exists$ dense open $W \subset \mathbb{A}_k^3$ s.t. $\forall P \in W(k) \cap \mathcal{P}_k$,
$\mathrm{sp}(e; P) \in \mathrm{Br}(U_P)/\mathrm{Br}(k)$ is its generator.

Additional assumption

In the following, we must assume k is non-2-closed. Then we assure that $W(k) \cap \mathcal{P}_k \neq \emptyset$ for all dense open set $W \subset \mathbb{A}_k^3$.

An existence result

- $\mathcal{O}_F = k[C, D], \quad F = k(C, D)$.
- $\mathcal{V}$: a 2-parameter family of affine diagonal quadrics defined as

$$\{x^2 - y^2 - Cz^2 + D = 0\} \text{ over } \mathcal{O}_F .$$

- $V = \{x^2 - y^2 - Cz^2 + D = 0\}$ over F.
- In this setting, we can also define the domain $\mathcal{P}_k$ and specializations $\mathrm{sp}(\cdot;\cdot)$.
- $e := \{CD, x + y\}_2 \in \mathrm{Br}(F(V))$.
- Recall $\mathrm{Br}(V) \subset \mathrm{Br}(F(V))$.

Proposition (U-.)

1. *e is in $\mathrm{Br}(V)$.*
2. *e is a uniform generator.*

Main Result

Recall:

- $\mathcal{O}_F = k[B, C, D], \quad F = k(B, C, D)$.
- $\mathcal{U}$: a 3-parameter family of affine diagonal quadrics defined as

$$\{x^2 + By^2 + Cz^2 + D = 0\} \text{ over } \mathcal{O}_F.$$

- $U = \{x^2 + By^2 + Cz^2 + D = 0\}$ over F.

Theorem (U-.)

For this 3-parameter family, there is no uniform generator.

Some comments

- To prove the theorem, it is essential to prove
$$\mathbf{Br}(V)/\mathbf{Br}(F) = 0.$$
We have done this by lengthy computation...
(by proving $d^{1,1}\colon H^1(G_F, \mathrm{Pic}(\overline{V})) \to H^3(G_F, \overline{F}^*)$ is not zero)
Can we find another strategy?
- Our results tell us the non-existence of such uniform generators implicitly relates the complexity of a given family.
- How about other class of surfaces?
-we've already done in the case of (projective) diagonal cubic surfaces.
- Similar problem for unramified cohomology?
-group structure?
-generator?

Thank you for your attention!