Zero-cycles on diagonal cubic surfaces over p-adic fields and their Brauer groups

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Definition (Chow group of zero-cycles on a variety)

For a variety X, define:

- the group of zero-cycles on X =
  free abelian group generated by 0-dimensional points on X:
  $$Z_0(X) := \mathbb{Z}[X_{(0)}]$$
- the 0-th Chow group of X:
  $$CH_0(X) := Z_0(X)/\sim_{rat},$$
  where $\sim_{rat}$ is the rational equivalence on $Z_0(X)$, i.e.
  $$Z \sim_{rat} Z' \overset{def}{\Leftrightarrow}$$
  $$\exists \eta_i \in X_{(1)}, \exists f_i \in \kappa(\eta_i)^* \text{ s.t. } \sum_i \mathrm{div}(f_i) = Z - Z'.$$
- Put $A_0(X) := \mathrm{Ker}(\deg \colon CH_0(X) \to \mathbb{Z})$.

Definition (Brauer group of a variety)

For a variety X over k, define:

- the Brauer group of X:

$$\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m).$$

- we have the following filtration:

$$\mathrm{Br}_0(X) \subset \mathrm{Br}_1(X) \subset \mathrm{Br}(X),$$

  where

$$\mathrm{Br}_0(X) := \mathrm{Im}(\mathrm{Br}(k) \to \mathrm{Br}(X)),$$
$$\mathrm{Br}_1(X) := \mathrm{Ker}(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})) \text{ (algebraic part).}$$

- Put $\mathrm{Br}(X)/\mathrm{Br}(k) = \mathrm{Br}(X)/\mathrm{Br}_0(X)$.

Remark

- the algebraic part $\mathrm{Br}_1(X)$ can be studied by the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{Gal}}(k, H^q_{\text{ét}}(\overline{X}, \mathbb{G}_m)) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathbb{G}_m).$$

- However, the so-called transcendental Brauer group $\mathrm{Br}(X)/\,\mathrm{Br}_1(X)$ is relatively hard to compute...
- Brauer groups has some applications to:
  - computation of $CH_0(X)$ (today's topic!).
  - Hasse principle for rational points or zero-cycles on X (if k is a global field).
  - etc....
- However, the following fact is known: If $\mathrm{ch}\, k = 0$ and X is a smooth complete intersection in $\mathbb{P}^n_k$ with $\dim X \geq 3$, then $\mathrm{Br}_0(X) = \mathrm{Br}(X)$.

Definition (Diagonal cubic surfaces)

Fix a field k. A diagonal cubic surface X over k is a smooth projective surface defined by

$$X = \mathbf{Proj}\, k[x, y, z, t]/(ax^3 + by^3 + cz^3 + dt^3),$$

where $a, b, c, d \in k^*$.

Remark

- diagonal cubic surfaces are rational.
  $\rightsquigarrow$ (relatively) easy to calculate their Brauer groups.

Theorem (Colliot-Thélène–Saito, Saito-Sato.)

- k : *a p-adic field.*
- $d \in k^* \setminus (k^*)^3$.
- X : *the d. c. s. over k defined by* $x^3 + y^3 + z^3 + dt^3 = 0$.

(1) *If* $p \neq 3$, *we have*

$$A_0(X) \cong \begin{cases} 0 & \textit{if } \operatorname{ord}(d) \equiv 0 \mod 3, \\ \mathbb{Z}/3\mathbb{Z} & \textit{if } \operatorname{ord}(d) \not\equiv 0 \mod 3 \textit{ and } \zeta \notin k, \\ (\mathbb{Z}/3\mathbb{Z})^2 & \textit{if } \operatorname{ord}(d) \not\equiv 0 \mod 3 \textit{ and } \zeta \in k. \end{cases}$$

(2) *If* $p = 3$, $\zeta = \zeta_3 \in k$ *and* $\operatorname{ord}(d) \equiv 1 \mod 3$, *then*

$$A_0(X) \cong (\mathbb{Z}/3\mathbb{Z})^2.$$

Question 1 (about structure)

If $p = 3$, How about $\operatorname{ord}(d) \neq 1$ cases?

Theorem 1 (U-.)

- k: a 3-adic field, $\zeta \in k$
- $d \in k^* \setminus (k^*)^3$.
- X: the d. c. s. over k defined by $x^3 + y^3 + z^3 + dt^3 = 0$.

Assume moreover:

- $\operatorname{ord}(d) \equiv 2 \mod 3$.
- the absolute ramification index of k is greater than 3.

Then we have

$$A_0(X) \cong (\mathbb{Z}/3\mathbb{Z})^2.$$

Question 2 (about structure)

How about $X : x^3 + y^3 + cz^3 + dt^3 = 0$ case?

Theorem 2. (U-.)

- k : a p-adic field.
- $c, d \in k^* \setminus (k^*)$ with $cd, c/d \notin (k^*)^3$.
- X : the d. c. s. over k defined by $x^3 + y^3 + cz^3 + dt^3 = 0$.

(1) If $p \neq 3$, we have

$$A_0(X) = \begin{cases} 0 & \text{if } \operatorname{ord}(c) \equiv \operatorname{ord}(d) \equiv 0 \mod 3, \\ \mathbb{Z}/3\mathbb{Z} & \text{otherwise.} \end{cases}$$

(2) If $p = 3$, $\zeta \in k$, $\operatorname{ord}(c-1)$ is greater then the absolute ramification index of k and $\operatorname{ord}(d) \equiv 1 \mod 3$, then

$$A_0(X) \cong \mathbb{Z}/3\mathbb{Z}.$$

Question 3 (about generator)

Which zero-cycles generate $CH_0(X)$?

Theorem 3 (U-.)

- k: a p-adic field. $\zeta \in k^*$
- $p \neq 3$.
- $d \in k^* \setminus (k^*)^3$.
- X: the d. c. s. over k defined by $x^3 + y^3 + z^3 + dt^3 = 0$.

Then $\mathrm{CH}_0(X)$ is generated by *the classes of rational points*.

Outline

- k: a p-adic field.
- X: a smooth projective variety over k.

Definition (Brauer-Manin pairing)

- Manin defined the following pairing

$$\langle \cdot, \cdot \rangle\colon \operatorname{Br}(X) \times Z_0(X) \to \mathbb{Q}/\mathbb{Z}$$

given by

$$\langle \alpha, \sum n_i[P_i] \rangle = \sum n_i \operatorname{inv}_k \operatorname{cores}_{\kappa(P_i)/k} \alpha(P_i).$$

- This pairing induces the following pairings

$$\langle \cdot, \cdot \rangle\colon \operatorname{Br}(X) \times CH_0(X) \to \mathbb{Q}/\mathbb{Z}$$

and

$$\langle \cdot, \cdot \rangle\colon \operatorname{Br}(X)/\operatorname{Br}(k) \times A_0(X) \to \mathbb{Q}/\mathbb{Z}$$

This pairing induces a map:

$$\phi_X : A_0(X) \to \mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(k), \mathbb{Q}/\mathbb{Z}).$$

About this ϕ_X, we know the following:

Theorem (Colliot-Thélène.)

If X is a rational surface, then the map ϕ_X is injective.

Theorem (Saito-Sato.)

- $\mathcal{U}$: *regular in codimension one, faithfully flat over* $\mathcal{O}_k$.
- $X := \mathcal{U} \times_{\mathcal{O}_k} k$*: smooth over* k, $Y := \mathcal{U} \times_{\mathcal{O}_k} \mathbb{F}$
- η : *a generic point of* Y
- $A_\eta := \mathcal{O}^h_{\mathcal{U},\eta}$, K_η : *the fractional field of* A_η.
- $\iota : \mathrm{Br}(X) \to \mathrm{Br}(K_\eta)$.

Assume $\iota^{-1}(\mathrm{Br}(A_\eta)) \subset \mathrm{Br}_0(X)$. *Then* ϕ_X *is surjective.*

Now we recall the definition of norm residue maps.

- Assume k contains the group μ_n (n-th roots of unity).
- Consider the composite of the following maps:

$$\begin{aligned}\{\cdot,\cdot\}_n : k^* \otimes k^* &\to H^1(k,\mu_n)\otimes H^1(k,\mu_n)\\ &\overset{\cup}{\to} H^2(k,\mu_n^{\otimes 2})\\ &\cong H^2(k,\mu_n) = {}_n\mathrm{Br}(k)\end{aligned}$$

Definition (Milnor K-group)

$$K_2^M(k) := k^* \otimes k^* / \langle x \otimes (-x), x \otimes (1-x)\rangle.$$

Definition (Norm residue map)

The above map factors through the Milnor K-group of k and defines the n-th norm residue map

$$\{\cdot,\cdot\}_n : K_2^M(k) \to {}_n\mathrm{Br}(k).$$

Expression of $\mathrm{Br}(X)$ by symbols

Fact

- X: a smooth, integral variety over k.
- $k(X)$: the function field of X

$\Rightarrow$ We have a canonical inclusion $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k(X))$.

Theorem (Merkurjev-Suslin, 1983)

- k: *a field with* $\mathrm{ch}(k) = 0$ *and containing* μ_3*.*

The third norm residue map induces the following:

$$K_2^M(k)/3 \overset{\cong}{\rightarrow} {}_3\mathrm{Br}(k)$$

$\Rightarrow$ We can represent ${}_3\mathrm{Br}(X) \subset {}_3\mathrm{Br}(k(X))$ by norm residue symbols.

Theorem (Manin.)

- $X : x^3 + y^3 + z^3 + dt^3 = 0$.
- $d \notin (k^*)^3$.

Then,

- $\mathrm{Br}(X)/\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.
- $\mathrm{Br}(X)/\mathrm{Br}(k)$ *is generated by the following two symbols*

$$e_1 = \left\{d, \frac{x+\zeta y}{x+y}\right\}_3, \quad e_2 = \left\{d, \frac{x+z}{x+y}\right\}_3$$

Theorem (Colliot-Thélène–Kanevsky–Sansuc, U.)

- $X : x^3 + y^3 + cz^3 + dt^3 = 0$.
- c, d, c/d, $cd \notin (k^*)^3$.

Then,

- $\mathrm{Br}(X)/\,\mathrm{Br}(k) \cong \mathbb{Z}\,/3\,\mathbb{Z}$.
- $\mathrm{Br}(X)/\,\mathrm{Br}(k)$ *is generated by the following symbol*

$$e_1 = \left\{\frac{d}{c}, \frac{x+\zeta y}{x+y}\right\}_3 .$$

Remark

An essestially same result is appeared in a recent paper written by Colliot-Thélène and Wittenberg.

Remark

In the case of $X : x^3 + by^3 + cz^3 + dt^3 = 0$,

- $\mathrm{Br}(X)/\mathrm{Br}(k) \cong 0$ or $\mathbb{Z}/3\mathbb{Z}$.
  (If k is a number field or a p-adic field, we have $\mathrm{Br}(X)/\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}$.
- we have no “uniform” generator in some sense.

Recall:

Theorem 1 (U-.)

- k : a 3-adic field. $\zeta \in k$
- $d \in k^* \setminus (k^*)^3$.
- X : the d. c. s. over k defined by $x^3 + y^3 + z^3 + dt^3 = 0$.

Assume moreover:

- $\mathrm{ord}(d) \equiv 2 \mod 3$.
- the absolute ramification index of k is greater than 3.

Then we have

$$A_0(X) \cong (\mathbb{Z}/3\mathbb{Z})^2.$$

First we briefly review on the proof of the following theorem:

Theorem (Saito-Sato.)

- k: *a* p*-adic field.*
- $d \in k^* \setminus (k^*)^3$.
- X: *the d. c. s. over* k *defined by* $x^3 + y^3 + z^3 + dt^3 = 0$.

(2) *If* $p = 3$, $\zeta = \zeta_3 \in k$ *and* $\operatorname{ord}(d) \equiv 1 \mod 3$, *then*

$$A_0(X) \cong (\mathbb{Z}/3\,\mathbb{Z})^2 (= \mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(k), \mathbb{Q}/\mathbb{Z})).$$

We have to prove that there exist a pair $(\mathcal{U}, \eta)$ such that

- $\mathcal{U}$: regular in codimension one, faithfully flat over $\mathcal{O}_k$.
- $X \cong \mathcal{U} \times_{\mathcal{O}_k} k$.
- η : a generic point of $Y := \mathcal{U} \times_{\mathcal{O}_k} \mathbb{F}$.
- $A_\eta := \mathcal{O}^h_{\mathcal{U},\eta}$, K_η : the fractional field of A_η.
- $\iota : \mathrm{Br}(X) \to \mathrm{Br}(K_\eta)$.
- $\iota^{-1}(\mathrm{Br}(A_\eta)) \subset \mathrm{Br}_0(X)$.

- We take as $\mathcal{U}$:

$$\mathcal{U} = \mathbf{Proj}\,\mathcal{O}_k[x,y,z,t]/(x^3+y^3+z^3+dt^3).$$

- η : the generic point of $Y := \mathcal{U} \times_{\mathcal{O}_k} \mathbb{F}$.

Definition

- We introduce a descending filtration U^m on K_η^*:

$$U^m K_\eta^* = \begin{cases} K_\eta^* & m = 0, \\ \{1 + x \mid \mathrm{ord}_{K_\eta}(x) \geq m\} & m \geq 1. \end{cases}$$

- We also define a descending filtration U^m on $H = H^2(K_\eta, \mu_3^{\otimes 2}) \cong {}_3\mathrm{Br}(K_\eta)$:

$$U^m H = \{U^m K_\eta^*, K_\eta^*\}_3.$$

fact

We have

$$_3\mathrm{Br}(A_\eta) \subset U^{3e'}H,$$

where $e = \mathrm{ord}_k(3), e' = \frac{3}{2}e$.

This fact immediately implies:

Lemma

If we have

- $e_1 = \left\{d, \frac{x+\zeta y}{x+y}\right\} \in U^{3e'-4}H$ and $\neq 0$ in $\mathrm{gr}_U^{3e'-4}H$,
- $e_2 = \left\{d, \frac{x+z}{x+y}\right\} \in U^{3e-4}H$ and $\neq 0$ in $\mathrm{gr}_U^{3e-4}H$,

Then we have $\iota^{-1}(\mathrm{Br}(A_\eta)) \subset \mathrm{Br}_0(X)$.

- Since the structure of $\mathrm{gr}_U^\bullet H$ is well-known by Bloch-Kato, the proof of the theorem almost reduces to the calculation of Milnor-K-theoretic symbols.
- However, this calculation is very hard...

Now we look at Theorem 1. Recall again:

Theorem 1 (U-.)

- k: a 3-adic field. $\zeta \in k$
- $d \in k^* \setminus (k^*)^3$.
- X: the d. c. s. over k defined by $x^3 + y^3 + z^3 + dt^3 = 0$.

Assume moreover:

- $\mathrm{ord}(d) \equiv 2 \mod 3$.
- the absolute ramification index of k is greater than 3.

Then we have $A_0(X) \cong (\mathbb{Z}/3\mathbb{Z})^2$.

Key of the proof of Theorem 1

- The strategy is basically same as the above theorem of Saito-Sato.
- We cannot take $x^3 + y^3 + z^3 + dt^3 = 0$ over $\mathcal{O}_k$ as $\mathcal{U}$, because A_η is not regular in this case.
  $\rightsquigarrow$ We have to take another model $\mathcal{U}$ and an appropriate choice of η.
- In this case, symbolic calculation is difficult, too.
  $\rightsquigarrow$ We have to leave an additional assumption $e > 3$...

Recall:

Theorem 2. (U-.)

- k: a p-adic field
- $c, d \in k^* \setminus (k^*)$ with $cd, c/d \notin (k^*)^3$.
- X: the d. c. s. over k defined by $x^3 + y^3 + cz^3 + dt^3 = 0$.

(1) If $p \neq 3$, we have

$$A_0(X) = \begin{cases} 0 & \text{if } \operatorname{ord}(c) \equiv \operatorname{ord}(d) \equiv 0 \mod 3, \\ \mathbb{Z}/3\mathbb{Z} & \text{otherwise.} \end{cases}$$

(2) If $p = 3$, $\zeta \in k$, $\operatorname{ord}(c-1)$ is greater then the absolute ramification index of k and $\operatorname{ord}(d) \equiv 1 \mod 3$, then

$$A_0(X) \cong \mathbb{Z}/3\mathbb{Z}.$$

Key of the proof of Theorem 2 (1)

- Our goal is
$$\iota^{-1}(\mathrm{Br}(A_\eta)) \subset \mathrm{Br}_0(X)$$
for some $(\mathcal{U}, \eta)$.
- Take
$$\mathcal{U} = \mathrm{Proj}\,\mathcal{O}_k[x, y, z, t]/(x^3 + y^3 + cz^3 + dt^3).$$
- divide the proof into 6 parts w.r.t $\mathrm{ord}_k(c)$ and $\mathrm{ord}_k(d)$.
- Instead of filtration argument, we use the following residual exact sequence:
$$0 \to {}_3\mathrm{Br}(A_\eta) \to {}_3\mathrm{Br}(K_\eta) \overset{\mathrm{res}_\eta}{\to} \kappa(\eta)^*/(\kappa(\eta)^*)^3.$$

Recall:

Theorem 3 (U-.)

- k: a p-adic field. $\zeta \in k^*$
- $p \neq 3$.
- $d \in k^* \setminus (k^*)^3$.
- X: the d. c. s. over k defined by $x^3 + y^3 + z^3 + dt^3 = 0$.

Then $\mathbf{CH_0}(X)$ is generated by *the classes of rational points*.

- It suffices to show $A_0(X)$ is generated by some differences of classes of rational points.
- May assume $\mathbf{ord}(d) \neq 0$.
- Put $f = \dfrac{x+\zeta y}{x+y}$, $g = \dfrac{x+z}{x+y}$.

Lemma

For a point $P \in X(k)$ s.t. $f(P) \in k^*$, the pairing $\langle [P], e_1\rangle \in \mathbb{Q}/\mathbb{Z}$ is considered as

$$(-1)^{\mathbf{ord}(d)\,\mathbf{ord}(f(P))} \cdot \overline{\left(\frac{f(P)^{\mathbf{ord}(d)}}{d^{\mathbf{ord}(f(P))}}\right)} \in \mathbb{F}^*/(\mathbb{F}^*)^3.$$

- We devide its proof into 3 parts:
  (i) $\zeta \notin (k^*)^3$.
  (ii) $\zeta \in (k^*)^3$, $1-\zeta \notin (k^*)^3$.
  (iii) $\zeta, 1-\zeta \in (k^*)^3$.

Key to the proof of (i): $\zeta \notin (k^*)^3$

- We can find explicit zero-cycles.
- Put

$$C_1 := [0 : 1 : -1 : 0] - [1 : 0 : -\zeta : 0],$$
$$C_2 := [1 : 0 : -\zeta : 0] - [1 : 0 : -\zeta^2 : 0].$$

- Noting that $\zeta \notin (k^*)^3$, we easily see that C_1 and C_2 generates $A_0(X)$.

Key to the proof of (ii): $\zeta \in (k^*)^3$, $1-\zeta \notin (k^*)^3$

- First of all, the above C_1 is non-trivial in $A_0(X)$.
- However, it seems difficult to construct the other generator explicitly.
- Instead, using some ideas due to Colliot-Thélène and Madore, we can prove the following:

Proposition

Let $E/\,\mathbb{F}$ be a curve defined by $x^3+y^3+z^3=0$. Then there exists a point $P \in E(\mathbb{F})$ s.t. $f(P)$ is non-trivial in $\mathbb{F}^*\,/(\mathbb{F}^*)^3$.

- By Hensel's lemma, we can get a lift $\widetilde{P} \in \mathcal{E}(\mathcal{O}_k)$, where $\mathcal{E}\,/\,\mathcal{O}_k : x^3+y^3+z^3=0$.
- Then we see that the zero-cycle

$$C_3 := [\widetilde{P} : 0] - [0:1:-1:0]$$

is just what we need.

Proof of Proposition

- Consider a curve
$$\overline{C} := \{(x+y)u^3 - (x+\zeta y)v^3 = 0\} \subset \mathbb{P}^1_E,$$
where $[u : v]$ is the homogeneous coordinate of $\mathbb{P}^1_{\mathbb{F}}$.
- Let E' be the blowing-up of $\overline{C}$ along two points
$$(O, [1:0]), \quad (Q, [0:1]),$$
where $O = [1:-1:0], Q = [1:-\zeta^2:0] \in E(\mathbb{F})$.
- $\pi \colon E' \to E$, Fix $O' \in \pi^{-1}(O)(\mathbb{F})$.
$\rightsquigarrow$ $\pi \colon (E', O') \to (E, O)$ is $\mathbb{F}$-isogeny of elliptic curves of degree 3.
- By Using the dual isogeny $\hat{\pi} \colon E \to E'$ and the Weil pairing $e \colon E[3] \times E[3] \to \mu_3$, we have an isomorphism
$$w \colon \operatorname{Ker} \pi \overset{\cong}{\to} \mu_3; \quad w(P) = e(R, Q) \in \mu_3,$$
where $R \in E[3]$ is a point s.t. $\hat{\pi}(R) = P$.

- $0 \to \operatorname{Ker}\pi \to E' \to E \to 0$ (exact)
  $\rightsquigarrow E(\mathbb{F}) \xrightarrow{\delta} H^1(\mathbb{F}, \operatorname{Ker}\pi) \to H^1(\mathbb{F}, E')$ (exact)
- By the Hasse-Weil bound for a genus-g curve $C/\,\mathbb{F}_q$:

$$|\#C(\mathbb{F}_q) - (q+1)| \leq 2g\sqrt{q},$$

  we have $H^1(\mathbb{F}, E') = 0$.
  $\rightsquigarrow$ the map δ is surjective.
- $1 \to \mu_3 \to \overline{\mathbb{F}}^* \xrightarrow{3} \overline{\mathbb{F}}^* \to 1$ (exact)
  $\rightsquigarrow \partial\colon\ \mathbb{F}^*/(\mathbb{F}^*)^3 \overset{\cong}{\to} H^1(\mathbb{F}, \mu_3)$.
- We can prove that

$$\partial^{-1} \circ w \circ \delta(P) = \begin{cases} 1 & P = O, Q, \\ f(P) & \text{otherwise.} \end{cases}$$

Key to the proof of (iii): $\zeta, 1-\zeta \in (k^*)^3$

- In this case, it seems difficult to construct one generator of $A_0(X)$ explicitly.
- By using g in addition to f and change of coordinates, we can also prove a variant of the previous proposition:

Proposition

Let $E/\mathbb{F}$ be a curve defined by $x^3+y^3+z^3=0$. Then there exist points P_1, P_2, and $P_3 \in E(\mathbb{F})$ s.t. $g(P_1), (f/g)(P_2)$ and $(fg)(P_3)$ are non-trivial in $\mathbb{F}^*/(\mathbb{F}^*)^3$.

- By using these proposition, we can construct two zero-cycles which come from rational points and generate $A_0(X)$.

Remaining tasks are:

- Improve techniques of calculating symbols in order to remove additional conditions.
- Any other surjectivity criterion?
- How about $\boldsymbol{x^3 + by^3 + cz^3 + dt^3 = 0}$ case?
- Does Theorem 3 hold in the case of $\boldsymbol{p = 3}$?
- Relation to indices.
- Other varieties?
- etc...

Thank you for your attention!