# $p$ 進体上の対角的 $3$ 次曲面のゼロサイクルについて
# On zero-cycles on diagonal cubic surfaces over $p$-adic fields

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## Definition (Chow group of zero-cycles)

For a variety $X/k$, we define:

- the group of zero-cycles on $X$:

$$Z_0(X) := \bigoplus_{P:\text{closed pts.}} \mathbb{Z}[P].$$

- the $0$-th Chow group of $X$:

$$CH_0(X) := Z_0(X)/\sim_{rat},$$

  where $\sim_{rat}$ is the rational equivalence on $Z_0(X)$.

- degree zero part of $CH_0(X)$:

$$A_0(X) := \mathrm{Ker}(\deg\colon CH_0(X) \to \mathbb{Z}),$$

  where $\deg(\sum n_i[P_i]) = \sum n_i[\kappa(P_i) : k]$.

## Definition (Brauer group of a variety)

For a variety $X/k$, define:

- the Brauer group of $X$:

$$\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m).$$

- Put $\mathrm{Br}(X)/\mathrm{Br}(k) = \mathrm{Br}(X)/\mathrm{Im}(\mathrm{Br}(k) \to \mathrm{Br}(X))$.

## Definition (Diagonal cubic surfaces)

Fix a field $k$. A diagonal cubic surface $X$ over $k$ is a smooth projective surface defined by

$$X = \mathbf{Proj}\, k[x, y, z, t]/(ax^3 + by^3 + cz^3 + dt^3),$$

where $a, b, c, d \in k^*$.

## Theorem (Colliot-Thélène–Saito, Saito-Sato.)

- $k$ : *a* $p$*-adic field.*
- $d \in k^* \setminus (k^*)^3$.
- $X$ : *the d. c. s. over* $k$ *defined by* $x^3 + y^3 + z^3 + dt^3 = 0$.

(1) *If* $p \neq 3$, *we have*

$$A_0(X) \cong \begin{cases} 0 & \textit{if } \operatorname{ord}(d) \equiv 0 \mod 3, \\ \mathbb{Z}/3\mathbb{Z} & \textit{if } \operatorname{ord}(d) \not\equiv 0 \mod 3 \textit{ and } \zeta \notin k, \\ (\mathbb{Z}/3\mathbb{Z})^2 & \textit{if } \operatorname{ord}(d) \not\equiv 0 \mod 3 \textit{ and } \zeta \in k. \end{cases}$$

(2) *If* $p = 3$, $\zeta = \zeta_3 \in k$ *and* $\operatorname{ord}(d) \equiv 1 \mod 3$, *then*

$$A_0(X) \cong (\mathbb{Z}/3\mathbb{Z})^2.$$

### Question 1 (about structure)

If $p = 3$, How about $\mathrm{ord}(d) \not\equiv 1$ cases?

### Theorem 1 (U-.)

- $k$: a $3$-adic field, $\zeta \in k$
- $d \in k^* \setminus (k^*)^3$.
- $X$: the d. c. s. over $k$ defined by $x^3 + y^3 + z^3 + dt^3 = 0$.

Assume moreover:

- $\mathrm{ord}(d) \equiv 2 \mod 3$.
- the absolute ramification index of $k$ is greater than $3$.

Then we have

$$A_0(X) \cong (\mathbb{Z}/3\mathbb{Z})^2.$$

## Question 2 (about structure)

How about $X : x^3 + y^3 + cz^3 + dt^3 = 0$ case?

## Theorem 2. (U-.)

- $k$ : a $p$-adic field.
- $c, d \in k^* \setminus (k^*)$ with $cd, c/d \notin (k^*)^3$.
- $X$ : the d. c. s. over $k$ defined by $x^3 + y^3 + cz^3 + dt^3 = 0$.

(1) If $p \neq 3$, we have

$$A_0(X) = \begin{cases} 0 & \text{if } \operatorname{ord}(c) \equiv \operatorname{ord}(d) \equiv 0 \mod 3, \\ \mathbb{Z}/3\mathbb{Z} & \text{otherwise.} \end{cases}$$

(2) If $p = 3$, $\zeta \in k$, $\operatorname{ord}(c-1)$ is greater then the absolute ramification index of $k$ and $\operatorname{ord}(d) \equiv 1 \mod 3$, then

$$A_0(X) \cong \mathbb{Z}/3\mathbb{Z}.$$

**Question 3 (about generator)**

Which zero-cycles generate $CH_0(X)$?

**Theorem 3 (U-.)**

- $k$: a $p$-adic field. $\zeta \in k^*$
- $p \neq 3$.
- $d \in k^* \setminus (k^*)^3$.
- $X$: the d. c. s. over $k$ defined by $x^3 + y^3 + z^3 + dt^3 = 0$.

Then $\mathrm{CH}_0(X)$ is generated by *the classes of rational points*.

- $k$: a $p$-adic field.
- $X$: a smooth projective variety over $k$.

**Definition (Brauer-Manin pairing)**

- Manin defined the following pairing

$$\langle \cdot, \cdot \rangle \colon \operatorname{Br}(X) \times CH_0(X) \to \mathbb{Q}/\mathbb{Z}$$

given by

$$\langle \alpha, \sum n_i [P_i] \rangle = \sum n_i \operatorname{inv}_k \operatorname{cores}_{\kappa(P_i)/k} \alpha(P_i).$$

- This pairing induces a map:

$$\phi_X \colon A_0(X) \to \operatorname{Hom}(\operatorname{Br}(X)/\operatorname{Br}(k), \mathbb{Q}/\mathbb{Z}).$$

About this $\phi_X$, we know the following:

**Theorem (Colliot-Thélène.)**

*If $X$ is a rational surface, then the map $\phi_X$ is injective.*

**Theorem (Saito-Sato.)**

- $\mathcal{U}$ : *regular in codimension one, faithfully flat over* $\mathcal{O}_k$.
- $X := \mathcal{U} \times_{\mathcal{O}_k} k$: *smooth over* $k$, $Y := \mathcal{U} \times_{\mathcal{O}_k} \mathbb{F}$
- $\eta$ : *a generic point of* $Y$
- $A_\eta := \mathcal{O}^h_{\mathcal{U},\eta}$, $K_\eta$ : *the fractional field of* $A_\eta$.
- $\iota : \mathrm{Br}(X) \to \mathrm{Br}(K_\eta)$.

*Assume* $\iota^{-1}(\mathrm{Br}(A_\eta)) \subset \mathrm{Im}(\mathrm{Br}(k) \to \mathrm{Br}(X))$. *Then* $\phi_X$ *is surjective.*

Recall:

**Theorem 3 (U-.)**

- $k$: a $p$-adic field. $\zeta \in k^*$
- $p \neq 3$.
- $d \in k^* \setminus (k^*)^3$.
- $X$: the d. c. s. over $k$ defined by $x^3 + y^3 + z^3 + dt^3 = 0$.

Then $\mathbf{CH_0}(X)$ is generated by *the classes of rational points*.

- The most essential part of the proof is to construct a rational point which is nontrivial w.r.t the Brauer-Manin pairing.
- Using some ideas due to Colliot-Thélène and Madore, we can prove the following:

**Proposition**

Let $E/\mathbb{F}$ be a curve defined by $x^3 + y^3 + z^3 = 0$. Then there exists a point $P \in E(\mathbb{F})$ s.t. $\dfrac{x + \zeta y}{x + y}(P)$ is non-trivial in $\mathbb{F}^* / (\mathbb{F}^*)^3$.

- By Hensel's lemma, we can get a lift $\widetilde{P} \in \mathcal{E}(\mathcal{O}_k)$, where $\mathcal{E} / \mathcal{O}_k : x^3 + y^3 + z^3 = 0$.
- Then we see that the rational point $[\widetilde{P} : 0]$ is just what we need.

# Proof of Proposition

- Using a curve

$$\overline{C} := \{(x+y)u^3 - (x+\zeta y)v^3 = 0\} \subset \mathbb{P}^1_E,$$

  where $[u : v]$ is the homogeneous coordinate of $\mathbb{P}^1_{\mathbb{F}}$, we can construct an $\mathbb{F}$-isogeny $\pi\colon (E', O') \to (E, O)$ of degree $3$.
- We have $\mathrm{Ker}\,\pi \cong \mu_3$.
- $0 \to \mu_3 \to E' \to E \to 0$ (exact), $H^1(\mathbb{F}, E') = 0$ $\rightsquigarrow$ the map $\delta\colon E(\mathbb{F}) \to H^1(\mathbb{F}, \mu_3) \cong \mathbb{F}^* / (\mathbb{F}^*)^3$ is surjective.
- We can prove that

$$\delta(P) = \begin{cases} 1 & P = O, Q, \\ \dfrac{x+\zeta y}{x+y}(P) & \text{otherwise.} \end{cases}$$

## Theorem (Manin.)

- $X : x^3 + y^3 + z^3 + dt^3 = 0$. $d \notin (k^*)^3$. $\zeta \in k$

*Then,*

- $\mathrm{Br}(X)/\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.
- $\mathrm{Br}(X)/\mathrm{Br}(k)$ *is generated by the following two symbols*

$$e_1 = \left\{d, \frac{x+\zeta y}{x+y}\right\}_3, \quad e_2 = \left\{d, \frac{x+z}{x+y}\right\}_3.$$

- The symbol $\{\cdot,\cdot\}$ means the composite of

$$\begin{aligned} k(X)^* \otimes k(X)^* &\to H^1(k(X),\mu_3) \otimes H^1(k(X),\mu_3) \\ &\xrightarrow{\cup} H^2(k(X),\mu_3^{\otimes 2}) \\ &\cong H^2(k(X),\mu_3) = {}_3\mathrm{Br}(k(X)) \end{aligned}$$

- $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k(X))$

- We cannot take $x^3 + y^3 + z^3 + dt^3 = 0$ over $\mathcal{O}_k$ as $\mathcal{U}$, because $A_\eta$ is not regular in this case.
  $\rightsquigarrow$ We have to take another model $\mathcal{U}$ and an appropriate choice of $\eta$.
- We prove $\iota^{-1}(\mathrm{Br}(A_\eta)) \subset \mathrm{Br}_0(X)$ by using the same mathod as in Saito-Sato:
- Symbolic calculation is difficult.
  $\rightsquigarrow$ We have to leave an additional assumption $e > 3$...

Recall:

### Theorem 2. (U-.)

- $k$: a $p$-adic field
- $c, d \in k^* \setminus (k^*)$ with $cd, c/d \notin (k^*)^3$.
- $X$: the d. c. s. over $k$ defined by $x^3 + y^3 + cz^3 + dt^3 = 0$.

(1) If $p \neq 3$, we have

$$A_0(X) = \begin{cases} 0 & \text{if } \operatorname{ord}(c) \equiv \operatorname{ord}(d) \equiv 0 \mod 3, \\ \mathbb{Z}/3\mathbb{Z} & \text{otherwise.} \end{cases}$$

(2) If $p = 3$, $\zeta \in k$, $\operatorname{ord}(c-1)$ is greater then the absolute ramification index of $k$ and $\operatorname{ord}(d) \equiv 1 \mod 3$, then

$$A_0(X) \cong \mathbb{Z}/3\mathbb{Z}.$$

Instead of using the above theorem of Manin, we use the following

**Theorem (Colliot-Thélène–Kanevsky–Sansuc, U.)**

- $X : x^3 + y^3 + cz^3 + dt^3 = 0$.
- $c$, $d$, $c/d$, $cd \notin (k^*)^3$.

*Then,*

- $\mathrm{Br}(X)/\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}$.
- $\mathrm{Br}(X)/\mathrm{Br}(k)$ *is generated by the following symbol*

$$e_1 = \left\{\frac{d}{c}, \frac{x+\zeta y}{x+y}\right\}_3.$$

Thank you for your attention!