On the Brauer group of diagonal cubic surfaces

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August 10, 2012. / Kyushu University.
Settings

- $k$: a field with $\text{ch}(k) = 0$ and $\mu_3 \subset k^*$.
- $\zeta \in k$: a fixed primitive cubic root of unity.
- $X$: a smooth projective diagonal cubic surface over $k$ defined by a homogeneous equation:

$$x^3 + by^3 + cz^3 + dt^3 = 0,$$

where $b$, $c$ and $d \in k^*$.
- $\pi : X \to \text{Spec } k$: the structure morphism.
- Consider the group $\text{Br}(X)/\text{Br}(k) := \text{Br}(X)/\pi^* \text{Br}(k)$. 
A known result

**Theorem (Manin.)**

- $X : x^3 + y^3 + z^3 + dt^3 = 0$.
- $d \notin (k^*)^3$.

Then,

- $\text{Br}(X) / \text{Br}(k) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.
- *We can take*

\[ \left\{ d, \frac{x + \zeta y}{x + y} \right\}_3, \quad \left\{ d, \frac{x + z}{x + y} \right\}_3 \]

*as generators of this group.*
Results

Theorem (Theorem 1, U.)

- $X : x^3 + y^3 + cz^3 + dt^3 = 0$.
- $c, d, c/d, cd \notin (k^*)^3$.

Then,

- $\text{Br}(X) / \text{Br}(k) \cong \mathbb{Z} / 3\mathbb{Z}$.
- We can take
  \[
  \left\{ \frac{d}{c}, \frac{x + \zeta y}{x + y} \right\}_3
  \]
  as a "uniform" generator of this group.

Remark

An essentially same result is appeared in a recent paper written by Colliot-Thélène and Wittenberg.
Theorem (Theorem 2, U.)

- $X : x^3 + by^3 + cz^3 + dt^3 = 0.$
- $\dim_{F_3} k^* / (k^*)^3 \geq 2.$

Then,

- $\text{Br}(X) / \text{Br}(k) \cong \mathbb{Z} / 3 \mathbb{Z}$ generically.
- There is no “uniform” generator of this group.
Definition

- $k$: a field
- $CSA(k)$: the set of isomorphism classes of central simple algebras over $k$
- $\sim$: Morita equivalence relation on $CSA(k)$

**Definition (Brauer group of $k$)**

\[
Br(k) := CSA(k) / \sim
\]

- $Br(k)$ becomes an abelian group w.r.t. $\otimes$.
- Unit element: $[k]$
- Inverse element of $[A]$: $[A^{op}]$
Example

- \( \text{Br}(\mathbb{C}) = 0, \quad \text{Br}(\mathbb{F}_q) = 0. \)
- \( \text{Br}(k(C)) = 0 \) for \( k = \bar{k} \) and \( C \): curve over \( k \).
- \( \text{Br}(\mathbb{R}) = \{ [\mathbb{R}], [\mathbb{H}] \} \cong \mathbb{Z}/2\mathbb{Z}, \)
  \( \mathbb{H} \): Hamiltonian quaternion algebra over \( \mathbb{R} \).
- \( \text{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}. \)

Basic problems for \( \text{Br}(k) \)

- How compute?
  \( \rightarrow \text{Br}(k) \cong H^2(k, \bar{k}^*) \) (Galois cohomology).
- How represent (construct) elements?
  \( \rightarrow \) Norm residue symbols.

Now we recall the definition of norm residue maps (Next slide).
Assume $k$ contains the group $\mu_n$ ($n$-th roots of unity).

Consider the composite of the following maps:

\[
\{\cdot, \cdot\}_n : k^* \otimes k^* \to H^1(k, \mu_n) \otimes H^1(k, \mu_n)
\]
\[
\cup \to H^2(k, \mu_n \otimes 2)
\]
\[
\cong H^2(k, \mu_n) = nBr(k)
\]

**Definition (Milnor $K$-group)**

\[K_2^M(k) := k^* \otimes k^*/\langle x \otimes (-x), x \otimes (1 - x) \rangle.\]

**Definition (Norm residue map)**

The above map factors through the Milnor $K$-group of $k$ and defines the $n$-th norm residue map

\[
\{\cdot, \cdot\}_n : K_2^M(k) \to nBr(k).
\]
Background

- Brauer group of a field can be interpreted as a Galois cohomology group: \( \text{Br}(k) \cong H^2(k, \bar{k}^*) \)
- In 1960’s, Grothendieck and other mathematicians constructed the theory of
  - scheme: \( k \rightsquigarrow X \)
  - étale cohomology: \( H^*_\text{Gal} \rightsquigarrow H^*_\text{et} \)

\( \Rightarrow \) We can consider Brauer group of a scheme \( X \)!

**Definition ((cohomological) Brauer group)**

\[
\text{Br}(X) := H^2_{\text{et}}(X, \mathbb{G}_m)
\]

**Remark**

We can also generalize Brauer groups by using Azumaya algebras (”sheaf version of CSAs”).
Today’s goal

Basic problems for $\text{Br}(X)$

- How compute?
- How represent (construct) elements?

Today’s talk:

- $X$: $k$-smooth projective surface defined by $ax^3 + by^3 + cz^3 + dt^3 = 0$.
- Consider $\text{Br}(X)/\text{Br}(k)$.
  - Its structure.
  - Its representation by norm residue symbols.
Representability of $\text{Br}(X)$ by symbols

**Fact**

- $X$: a smooth, integral variety over $k$.
- $k(X)$: the function field of $X$

$\Rightarrow$ We have a canonical inclusion $\text{Br}(X) \hookrightarrow \text{Br}(k(X))$.

**Theorem (Merkurjev-Suslin, 1983)**

- $k$: a field with $\text{ch}(k) = 0$ and containing $\mu_3$.

The third norm residue map induces the following:

$$K_2^M(k)/3 \cong 3\text{Br}(k)$$

$\Rightarrow$ We can represent $3\text{Br}(X) \subset 3\text{Br}(k(X))$ by norm residue symbols.
Naive notion of specialization

- $X_{(a,b)}$: a family of varieties parametrized by $k^2$.

- $e = e(a, b)$: a certain structure on "$X_{(a,b)}$".

The specialization of $e$ at $P = (a_0, b_0)$ is:

$$\text{sp}(e; P) := e(a_0, b_0).$$
Definition of specialization

- \( \mathcal{O}_F = k[a_1, \ldots, a_r] \), \( \mathbb{A}^r_k = \text{Spec } \mathcal{O}_F \)
- \( \mathcal{F} \): the fractional field of \( \mathcal{O}_F \)
- \( f_1, \ldots, f_m \): homogeneous polynomials in \( \mathcal{O}_F[x_0, \ldots, x_n] \)
- \( \mathcal{X} = \text{Proj}(\mathcal{O}_F[x_0, \ldots, x_n]/(f_1, \ldots, f_m)) \)
- \( X_F = \mathcal{X} \times_{\mathbb{A}_k^r} \mathcal{F} \): generic fiber of \( \mathcal{X} \to \mathbb{A}_k^r \)
- \( e \in \text{Br}(X_F) \)

\[ \Rightarrow \exists S: \text{an affine open subset in } \mathbb{A}_k^r \text{ s.t.} \]

\[ e \text{ can be lifted to } \tilde{e} \in \text{Br}(\mathcal{X} \times_{\mathbb{A}_k^r} S). \]
Take $P : \text{Spec } k \to S$. Consider the following diagram:

$$
\begin{array}{ccc}
X_P & \xrightarrow{P} & X \times_{\mathbb{A}^r_k} S & \xleftarrow{\pi_S} & X_F \\
\downarrow{\pi_P} & & \downarrow{\pi_S} & & \downarrow{\pi_F} \\
\text{Spec } k & \xrightarrow{P} & S & \xleftarrow{\pi_F} & \text{Spec } F
\end{array}
$$

**Definition (Specialization)**

We define the specialization of $e$ at $P$ as

$$\text{sp}(e; P) := P^*\tilde{e} \in \text{Br}(X_P).$$
\(x^3 + y^3 + cz^3 + dt^3 = 0\) case

- \(\mathcal{O}_F = k[c, d],\ F = k(c, d)\).
- \(X_F = \text{Proj} F[x, y, z, t]/(x^3 + y^3 + cz^3 + dt^3)\).
- \(X_P = \text{Proj} k[x, y, z, t]/(x^3 + y^3 + c_0z^3 + d_0t^3)\) for \(P = (c_0, d_0) \in k^* \times k^*\).
- \(e_1 = \left\{ \frac{d}{c}, \frac{x + \zeta y}{x + y} \right\}_3 \in \text{Br}(X_F)\).

**Theorem (Theorem 1, U.)**

\(\forall P = (c_0, d_0) \in k^* \times k^*\) such that \(c_0, d_0, c_0d_0\) and \(d_0/c_0 \notin (k^*)^3\), \(\text{sp}(e_1; P)\) is a generator of the group \(\text{Br}(X_P)/\text{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}\).
$x^3 + by^3 + cz^3 + dt^3 = 0$ case

- $\mathcal{O}_F = k[b, c, d]$, $F = k(b, c, d)$.
- $X_F = \operatorname{Proj} F[x, y, z, t]/(x^3 + by^3 + cz^3 + dt^3)$.
- $X_P = \operatorname{Proj} k[x, y, z, t]/(x^3 + b_0y^3 + c_0z^3 + d_0t^3)$ for $P = (b_0, c_0, d_0) \in \mathbb{A}^3$.

**Definition ($\mathcal{P}_k$)**

$$\mathcal{P}_k := \{ P \in (k^*)^3 \mid \text{Br}(X_P)/\text{Br}(k) \cong \mathbb{Z}/3\mathbb{Z} \}.$$  

**Proposition**

The following conditions are equivalent:

- $\dim_{\mathbb{F}_3} k^* / (k^*)^3 \geq 2$.
- $\mathcal{P}_k$ is Zariski dense in $(\mathbb{G}_m)^3$. 
$x^3 + by^3 + cz^3 + dt^3 = 0$ case

- $\mathcal{O}_F$, $F$, $X_F$, $X_P$: as above.

**Theorem (Theorem2, U.)**

Assume $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$. Then there is no element $e \in \text{Br}(X_F)$ satisfying the following condition:

there exists a dense open subset $W \subset (\mathbb{G}_{m,k})^3$ such that

- $\text{sp}(e; \cdot)$ is defined on $W(k) \cap \mathcal{P}_k$;
- for all $P \in W(k) \cap \mathcal{P}_k$, $\text{sp}(e; P)$ is a generator of the group $\text{Br}(X_P)/\text{Br}(k)$. 

Tetsuya UEMATSU
On the Brauer group of diagonal cubic surfaces
An exact sequence

Let $X$ be a smooth projective diagonal cubic surface over $k$ defined by

$$x^3 + y^3 + cz^3 + dt^3 = 0.$$  

We start the following exact sequence:

$$0 \to \text{Br}_1(X)/\text{Br}(k) \to H^1(k, \text{Pic}(X)) \xrightarrow{\text{d}^{1,1}} H^3(k, k^*),$$

where

- $\overline{X} = X \times_k \bar{k}$.
- $\text{Pic}(\overline{X})$: Picard group of $\overline{X}$.
- $\text{Br}_1(X) := \text{Ker}(\text{Br}(X) \to \text{Br}(\overline{X}))$.  

Tetsuya UEMATSU  
On the Brauer group of diagonal cubic surfaces
For our $X$, we have

1. $X(k) \neq \emptyset$
   $\Rightarrow d^{1,1} = 0$
   $\Rightarrow \text{Br}_1(X)/\text{Br}(k) \cong H^1(k, \text{Pic}(\overline{X})).$

2. $X$ is (geometrically) rational
   $\Rightarrow \text{Br}(\overline{X}) = 0$
   $\Rightarrow \text{Br}_1(X) = \text{Br}(X)$.

$\therefore \text{Br}(X)/\text{Br}(k) \cong H^1(k, \text{Pic}(\overline{X}))$

$\Rightarrow$ We can describe the Brauer group in terms of divisors!
Divisors on $X$

- $\gamma := (d/c)^{1/3}$, $k' = k(\gamma)$, $G = \text{Gal}(k'/k)$.
- For $i = 0, 1, 2$, consider the following lines on $X$:

  $L(i) : x + \zeta^i y = z + \zeta^i \gamma t = 0$, 
  $L'(i) : x + \zeta^i y = z + \zeta^{i+1} \gamma t = 0$, 
  $L''(i) : x + \zeta^i y = z + \zeta^{i+2} \gamma t = 0$.

- $H$: hyperplane section defined by $\{x = 0\}$.
- $\mathcal{D} := \mathbb{Z} H \oplus \bigoplus_{i=0}^{2} \mathbb{Z} L(i) \oplus \bigoplus_{i=0}^{2} \mathbb{Z} L'(i) \oplus \bigoplus_{i=0}^{2} \mathbb{Z} L''(i)$.
- $\mathcal{D}_0 := \text{Ker}(\mathcal{D} \to \text{Pic}(X_{k'}))$. 
Proposition

1. We have an exact sequence:

\[ 0 \to D_0 \to D \to \text{Pic}(X_{k'}) \to 0 \]

2. We have the following isomorphisms:

\[
H^1(k, \text{Pic}(X)) \cong H^1(G, \text{Pic}(X_{k'})) \\
\cong \frac{D_0 \cap N_G D}{N_G D_0} \\
\cong \mathbb{Z}/3\mathbb{Z}.
\]
Put $\text{Br}(X_{k'}/X) := \text{Ker}(\text{Br}(X) \rightarrow \text{Br}(X_{k'}))$.

We have the following diagram with all rows and columns exact:

$$
\begin{array}{ccc}
\text{Br}(k'/k) & \longrightarrow & \text{Br}(k'/k) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{Br}(X_{k'}/X) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^2(G, k'(X)^*) \longrightarrow^{\text{div}} H^2(G, \text{Div}(X_{k'})) \\
\downarrow & & \downarrow \\
& & \\
& & \\
\end{array}
$$

By chasing this diagram, we get a symbolic generator of $\text{Br}(X)/\text{Br}(k)$. \qed
Recall the statement of Theorem 2.

- $\mathcal{O}_F = k[b, c, d]$, $F = k(b, c, d)$.
- $X_F = \text{Proj } F[x, y, z, t]/(x^3 + by^3 + cz^3 + dt^3)$.
- $X_P = \text{Proj } k[x, y, z, t]/(x^3 + b_0y^3 + c_0z^3 + d_0t^3)$ for $P = (b_0, c_0, d_0) \in k^* \times k^* \times k^*$.

**Theorem (Theorem 2, U.)**

Assume $\dim_{F_3} k^*/(k^*)^3 \geq 2$. Then there is no element $e \in \text{Br}(X_F)$ satisfying the following condition:

there exists a dense open subset $W \subset (\mathbb{G}_m, k)^3$ such that

- $\text{sp}(e; \cdot)$ is defined on $W(k) \cap \mathcal{P}_k$;
- for all $P \in W(k) \cap \mathcal{P}_k$, $\text{sp}(e; P)$ is a generator of the group $\text{Br}(X_P)/\text{Br}(k)$. 
A vanishing theorem

- $F = k(\lambda, \mu, \nu)$.
- $X = \text{Proj } F[x, y, z, t]/(x^3 + \lambda y^3 + \mu z^3 + \lambda \mu \nu t^3)$.

Theorem 2 is a consequence of the following theorem:

**Theorem**

$$\text{Br}(X)/\text{Br}(F) = 0.$$ 

So we are devoted to the proof of this theorem in the following argument.

- $\alpha = \lambda^{1/3}, \alpha' = \mu^{1/3}, \gamma = \nu^{1/3}$.
- $F' = F(\alpha, \gamma), F'' = F'(\alpha')$. 
Fact

We have an exact sequence:

\[ 0 \to \text{Br}(X)/\text{Br}(F) \to H^1(F, \text{Pic}(\overline{X})) \xrightarrow{d^{1,1}} H^3(F, \overline{F}^*) \]

and isomorphisms:

\[ H^1(F, \text{Pic}(\overline{X})) \cong H^1(F'/F, \text{Pic}(X_{F'})) \cong \mathbb{Z}/3\mathbb{Z}. \]

Our goal

To prove:

- a generator \( \phi \in H^1(F'/F, \text{Pic}(X_{F'})) \) does not vanish in \( H^3(F, \overline{F}^*) \).

In the following, we may assume \( k \) contains all roots of unity.
Goal and Strategy

Problem

\[ H^3(F, \overline{F}^*) \] is too big to check \( \phi \) is zero or not in it...

\[ \Rightarrow \text{So we have to choose a smaller (finite) group } G \text{ so that the image of } \phi \text{ in } G \text{ is nonzero.} \]

Strategy

- Step 1: Explicit description of \( d^{1,1} \).
- Step 2: Consider \( \phi \in H^3(F, \overline{F}^*) \) as an element of \( H^3(F''/F, \mu_3) \).
- Step 3: Reduction to \( H^2 \).
- Step 4: Reduction to \( H^1 \).
Step 1: Explicit description of $d^{1,1}$

- $0 \to \mathcal{D}_0 \to \mathcal{D} \to \text{Pic}(X_{F'}) \to 0$
- $0 \to F'^* \to \text{div}^{-1}(\mathcal{D}_0) \to \mathcal{D}_0 \to 0$

Proposition (Kresch-Tschinkel, 2006.)

We have the following commutative diagram.

\[
\begin{array}{ccc}
H^1(F'/F, \text{Pic}(X_{F'})) & \xrightarrow{\partial} & H^2(F'/F, \mathcal{D}_0) \\
& \downarrow d^{1,1}_{F'} & \downarrow \delta \\
& H^3(F'/F, F'^*) & \\
\end{array}
\]

Using this result, we get an explicit cocycle $\phi \in H^3(F'/F, (F')^*)$. 

Step 2: $\phi$ as an element of $H^3(F''/F, \mu_3)$

Using the following commutative diagram with exact rows:

$$
\begin{array}{cccc}
H^3(F'/F, F'\ast) & \xrightarrow{i_{F''}} & H^3(F''/F, F''\ast) & \xrightarrow{3} H^3(F''/F, (F''\ast)^3) \\
H^3(F''/F, \mu_3) & \xrightarrow{i_{F''}} & H^3(F, F^* ) & \xrightarrow{3} H^3(F, F^*), \\
H^3(F, \mu_3) & \xrightarrow{i_{\overline{F}}} & H^3(F, \overline{F}^*) & \xrightarrow{3} H^3(F, \overline{F}^*),
\end{array}
$$

We can consider $\phi$ as an element of $H^3(F''/F, \mu_3)$. 
D ⊂ \mathbb{A}^3_k: a principal divisor, \( k(D) \): its residue field.

For each \( D \), we have a comm. diagram:

\[
\begin{array}{cccccc}
H^3(F''/F, \mu_3) & \xrightarrow{i^{F''}_F} & H^3(F, \mu_3) & \xrightarrow{\text{res}_D} & H^2(k(D), \mathbb{Z}/3\mathbb{Z}) & \xrightarrow{\mathbb{Q}} & H^2(k(D), \mu_3) \\
& & \downarrow & & \downarrow & & \\
& & H^3(F, \mathbb{Q}/\mathbb{Z}(1)) & \xrightarrow{\text{res}_D} & H^2(k(D), \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\mathbb{Q}} & H^2(k(D), \mathbb{Q}/\mathbb{Z}(1)) \\
& & \downarrow \mathbb{Q} & & \downarrow \mathbb{Q} & & \\
& & H^3(F, \overline{F}^*) & & & H^2(k(D), \overline{k(D)}^*) \\
\end{array}
\]

Hence it suffices to prove the non-vanishingness of \( \phi \) in \( H^2(k(D), \mu_3) \) for some \( D \).
Step 3: Reduction to $H^2$

- Take $D = \{ \mu = 0 \} \cong \mathbb{A}^2_k$.
- $k(D) = k(\lambda, \nu)$.
- $k(D)' := k(D)(\alpha), \quad \alpha = \lambda^{1/3}$.
- We have the inflation map

$$\inf : H^2(k(D)'/k(D), \mu_3) \to H^2(k(D), \mu_3).$$

Lemma (Reduction to finite group)

There exists $r\phi \in H^2(k(D)'/k(D), \mu_3)$ such that

$$\text{res}_D(\phi) = \inf(r\phi).$$

This cocycle $r\phi$ is used in Step 4.
Step 4: Reduction to $H^1$

- For simplicity, put
  \[ E := k(D), \quad E' := k(D)', \quad \Phi = \text{res}_D(\phi). \]

- Put $D' := \{\nu = 0\} \subset \mathbb{A}^2_k = \text{Spec} \ k[\lambda, \nu]$.

- $k(D')(\cong k(\lambda))$: the residue field of $D'$.

- The residue map along to $D'$:
  \[ \text{res}_{D'} : H^2(E, \mu_3) \to H^1(k(D'), \mathbb{Z} / 3 \mathbb{Z}). \]

We consider the residue of $\Phi \in H^2(E, \mu_3)$ along $D'$. 
Step 4: Reduction to $H^1$

Lemma

- $\text{res}_{D'}(\Phi) \in H^1(k(D'), \mathbb{Z}/3\mathbb{Z})$ comes from $H^1(k(D')'/k(D'), \mathbb{Z}/3\mathbb{Z})$.

- The inflation $H^1(k(D')'/k(D'), \mathbb{Z}/3\mathbb{Z}) \to H^1(k(D'), \mathbb{Z}/3\mathbb{Z})$ is injective.

It's easy to check the nontriviality of $\phi \in H^1(k(D')'/k(D'), \mathbb{Z}/3\mathbb{Z})$. Therefore this complete the proof of Theorem 2. □
For more detail, see

Thank you for your attention! However, we have a few minutes and I want to talk an additional topic... As an application, we can compute Chow groups of some diagonal cubic surfaces, by using symbolic generators.
Algebraic cycles

- $X$: a variety over a field $k$.
- $X_{(r)}$: the set of irr. cl. subsch. of $X$ with dimension $r$.
- $Z_r(X) = \bigoplus_{D \in X_{(r)}} \mathbb{Z}[D]$: the group of $r$-cycles on $X$.
- $Z_r(X)_{rat} = \langle \text{div}(f) \mid f \in k(W)^*, \exists W \in X_{(r+1)} \rangle_{\mathbb{Z}}$

**Definition (Chow group)**

$$\text{CH}_r(X) := Z_r(X)/Z_r(X)_{rat}$$

**Definition (degree map)**

$$\deg: \text{CH}_0(X) \to \mathbb{Z}; \sum n_P[P] \mapsto \sum n_P[k(P) : k].$$

Put $A_0(X) := \text{Ker}(\deg: \text{CH}_0(X) \to \mathbb{Z})$. 
Brauer-Manin pairing

- \( k \): \( p \)-adic field (fin. ext. of \( \mathbb{Q}_p \)), \( \mathbb{F} \): its residue field.
- \( \text{inv}_k : \text{Br}(k) \rightarrow \mathbb{Q}/\mathbb{Z} \): invariant map of \( k \).
- \( X \): a variety over \( k \).

**Definition (Brauer-Manin pairing)**

\[
\langle \cdot, \cdot \rangle : \text{CH}_0(X) \times \text{Br}(X) \rightarrow \mathbb{Q}/\mathbb{Z};
\]

\[
\langle \sum n_P[P], \mathcal{A} \rangle = \sum n_P \text{inv}_k(\text{cores}_k(P)/k(P^*\mathcal{A})).
\]

This pairing induces a map:

\[
\phi_X : A_0(X) \rightarrow \text{Hom}(\text{Br}(X)/\text{Br}(k), \mathbb{Q}/\mathbb{Z}).
\]
Known results of $\phi_X$

**Theorem (Colliot-Thélène, 1983.)**

*If $X$ is a rational surface, then the map $\phi_X$ is injective.*

**Theorem (Saito-Sato, 2009.)**

- $\mathcal{U}$ is regular, faithfully flat over $\mathcal{O}_k$.
- $X := \mathcal{U} \times \mathcal{O}_k k$: smooth over $k$, $Y := \mathcal{U} \times \mathcal{O}_k \bar{F}$
- $\eta$: a generic point of $Y$
- $A_\eta := \mathcal{O}_{\mathcal{U}, \eta}$, $K_\eta$: the fractional field of $A_\eta$.
- $\iota : \text{Br}(X) \to \text{Br}(K_\eta)$.

Assume $\iota^{-1}(\text{Br}(A_\eta)) = 0$. Then $\phi_X$ is surjective.
If we can confirm the condition $\nu^{-1}(Br(A_\eta)) = 0$ by using the symbolic representation of $Br(X)$, we decide the structure of $A_0(X)$.

- $k$: $p$-adic field, $\text{ord}_k$: the normalized valuation.

**Theorem (Saito-Sato, 2009.)**

Let $X = \text{Proj} \ k[x, y, z, t]/(x^3 + y^3 + z^3 + dt^3)$.

1. Assume $p \neq 3$. Then

$$A_0(X) = \begin{cases} 
0 & \text{if } \text{ord}_k(d) \equiv 0(3), \\
\mathbb{Z}/3\mathbb{Z} & \text{if } \text{ord}_k(d) \not\equiv 0 \text{ and } \mu_3 \not\subset k, \\
(\mathbb{Z}/3\mathbb{Z})^2 & \text{if } \text{ord}_k(d) \not\equiv 0 \text{ and } \mu_3 \subset k.
\end{cases}$$

2. Assume $p = 3$, $\mu_3 \subset k$ and $\text{ord}(d) \equiv 1(3)$. Then

$$A_0(X) = (\mathbb{Z}/3\mathbb{Z})^2.$$
Theorem (U.)

Let $X = \text{Proj } k[x, y, z, t]/(x^3 + y^3 + cz^3 + dt^3)$ with $c, d, cd$ and $d/c \notin (k^*)^3$.

1. Assume $p \neq 3$. Then

$$A_0(X) = \begin{cases} 
0 & \text{if } \text{ord}_k(c) \equiv \text{ord}_k(d) \equiv 0(3), \\
\mathbb{Z}/3\mathbb{Z} & \text{otherwise}.
\end{cases}$$

2. Assume $p = 3$, $\mu_3 \subset k$ and

$\text{ord}_k(c) = 0$, $\text{ord}_k((c - 1)/3) > 0$, $\text{ord}_k(d) \equiv 1(3)$.

Then

$$A_0(X) = \mathbb{Z}/3\mathbb{Z}.$$
Thank you for your attention!