On the Brauer group of diagonal cubic surfaces

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Settings

- k: a field with $\mathrm{ch}(k) = 0$ and $\mu_3 \subset k^*$.
- $\zeta \in k$: a fixed primitive cubic root of unity.
- X: a smooth projective diagonal cubic surface over k defined by a homogeneous equation:

$$x^3 + by^3 + cz^3 + dt^3 = 0,$$

  where b, c and $d \in k^*$.
- $\pi : X \to \operatorname{Spec} k$: the structure morphism.
- Consider the group $\mathrm{Br}(X)/\,\mathrm{Br}(k) := \mathrm{Br}(X)/\pi^* \,\mathrm{Br}(k)$.

A known result

Theorem (Manin.)

- $X : x^3 + y^3 + z^3 + dt^3 = 0$.
- $d \notin (k^*)^3$.

Then,

- $\mathrm{Br}(X)/\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.
- *We can take*

$$\left\{d, \frac{x+\zeta y}{x+y}\right\}_3, \quad \left\{d, \frac{x+z}{x+y}\right\}_3$$

as generators of this group.

Results

Theorem (Theorem 1, U.)

- $X : x^3 + y^3 + cz^3 + dt^3 = 0$.
- c, d, c/d, $cd \notin (k^*)^3$.

Then,

- $\mathrm{Br}(X)/\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}$.
- *We can take*

$$\left\{ \frac{d}{c}, \frac{x+\zeta y}{x+y} \right\}_3$$

as a "uniform" generator of this group.

Remark

An essestially same result is appeared in a recent paper written by Colliot-Thélène and Wittenberg.

Results

Theorem (Theorem 2, U.)

- $X : x^3 + by^3 + cz^3 + dt^3 = 0$.
- $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$.

Then,

- $\mathrm{Br}(X)/\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}$ *generically.*
- *There is no "uniform" generator of this group.*

Definition

- k: a field
- $CSA(k)$: the set of isomorphism classes of central simple algebras over k
- $\sim$: Morita equivalence relation on $CSA(k)$

Definition (Brauer group of k)

$$\mathrm{Br}(k) := CSA(k)/\sim$$

- $\mathrm{Br}(k)$ becomes an abelian group w.r.t. $\otimes$.
- unit element: $[k]$
- inverse element of $[A]$: $[A^{op}]$

Example

- $\mathrm{Br}(\mathbb{C}) = 0, \quad \mathrm{Br}(\mathbb{F}_q) = 0.$
- $\mathrm{Br}(k(C)) = 0$ for $k = \overline{k}$ and C: curve over k.
- $\mathrm{Br}(\mathbb{R}) = \{[\mathbb{R}], [\mathbb{H}]\} \cong \mathbb{Z}/2\mathbb{Z}$,
  $\mathbb{H}$: Hamiltonian quaternion algebra over $\mathbb{R}$.
- $\mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$.

Basic problems for $\mathrm{Br}(k)$

- How compute?
  $\longrightarrow \mathrm{Br}(k) \cong H^2(k, \overline{k}^*)$ (Galois cohomology).
- How represent (construct) elements?
  $\longrightarrow$ Norm residue symbols.

Now we recall the definition of norm residue maps (Next slide).

- Assume k contains the group μ_n (n-th roots of unity).
- Consider the composite of the following maps:

$$\begin{aligned}\{\cdot,\cdot\}_n : k^* \otimes k^* &\longrightarrow H^1(k,\mu_n) \otimes H^1(k,\mu_n)\\ &\xrightarrow{\cup} H^2(k,\mu_n^{\otimes 2})\\ &\cong H^2(k,\mu_n) = {}_n\mathrm{Br}(k)\end{aligned}$$

Definition (Milnor K-group)

$$K_2^M(k) := k^* \otimes k^* / \langle x \otimes (-x), x \otimes (1-x) \rangle.$$

Definition (Norm residue map)

The above map factors through the Milnor K-group of k and defines the n-th norm residue map

$$\{\cdot,\cdot\}_n : K_2^M(k) \longrightarrow {}_n\mathrm{Br}(k).$$

Background

- Brauer group of a field can be interpreted as a Galois cohomology group: $\mathrm{Br}(k) \cong H^2(k, \overline{k}^*)$
- In 1960's, Grothendieck and other mathematicans constructed the theory of
  - scheme: $k \rightsquigarrow X$
  - étale cohomology: $H^*_{\mathrm{Gal}} \rightsquigarrow H^*_{et}$

$\Rightarrow$ We can consider Brauer group of a scheme X!

Definition ((cohomological) Brauer group)

$$\mathrm{Br}(X) := H^2_{\mathrm{et}}(X, \mathbb{G}_m)$$

Remark

We can also generalize Brauer groups by using Azumaya algebras("sheaf version of CSAs").

Today's goal

Basic problems for $\mathrm{Br}(X)$

- How compute?
- How represent (construct) elements?

Today's talk:

- X: k-smooth projective surface defined by $ax^3 + by^3 + cz^3 + dt^3 = 0$.
- Consider $\mathrm{Br}(X)/\mathrm{Br}(k)$.
    - Its structure.
    - Its representation by norm residue symbols.

Representability of $\mathrm{Br}(X)$ by symbols

Fact

- X: a smooth, integral variety over k.
- $k(X)$: the function field of X

$\Rightarrow$ We have a canonical inclusion $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k(X))$.

Theorem (Merkurjev-Suslin, 1983)

- k: *a field with* $\mathrm{ch}(k) = 0$ *and containing* μ_3.

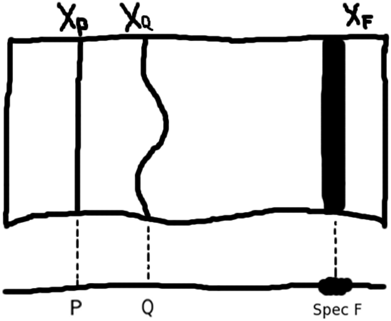
The third norm residue map induces the following:

$$K_2^M(k)/3 \xrightarrow{\cong} {}_3\mathrm{Br}(k)$$

$\Rightarrow$ We can represent ${}_3\mathrm{Br}(X) \subset {}_3\mathrm{Br}(k(X))$ by norm residue symbols.

Naive notion of specialization

- $X_{(a,b)}$: a family of varieties parametrized by k^2.



- $e = e(a,b)$: a certain structure on "$X_{(a,b)}$".

The specialization of e at $P = (a_0, b_0)$ is:

$$\mathrm{sp}(e; P) := e(a_0, b_0).$$

Definition of specialization

- $\mathcal{O}_F = k[a_1, \dots, a_r]$, $\mathbb{A}_k^r = \operatorname{Spec} \mathcal{O}_F$
- F: the fractional field of $\mathcal{O}_F$
- $f_1, \dots, f_m$: homogeneous polynomials in $\mathcal{O}_F[x_0, \dots, x_n]$
- $\mathcal{X} = \operatorname{Proj}(\mathcal{O}_F[x_0, \dots, x_n]/(f_1, \dots, f_m))$
- $X_F = \mathcal{X} \times_{\mathbb{A}_k^r} F$: generic fiber of $\mathcal{X} \to \mathbb{A}_k^r$
- $e \in \operatorname{Br}(X_F)$

$\Rightarrow \exists S$: an affine open subset in $\mathbb{A}_k^r$ s.t.

$$e \text{ can be lifted to } \widetilde{e} \in \operatorname{Br}(\mathcal{X} \times_{\mathbb{A}_k^r} S).$$

Definition of specialization

Take $P : \mathrm{Spec}\, k \longrightarrow S$. Consider the following diagram:

$$\begin{array}{ccccc} X_P & \xrightarrow{P} & \mathcal{X} \times_{\mathbb{A}^r_k} S & \longleftarrow & X_F \\ \downarrow \pi_P & \square & \downarrow \pi_S & \square & \downarrow \pi_F \\ \mathrm{Spec}\, k & \xrightarrow{P} & S & \longleftarrow & \mathrm{Spec}\, F \end{array}$$

Definition (Specialization)

We define the specialization of e at P as

$$\mathrm{sp}(e; P) := P^* \widetilde{e} \in \mathrm{Br}(X_P).$$

$x^3 + y^3 + cz^3 + dt^3 = 0$ case

- $\mathcal{O}_F = k[c, d]$, $F = k(c, d)$.
- $X_F = \operatorname{Proj} F[x, y, z, t]/(x^3 + y^3 + cz^3 + dt^3)$.
- $X_P = \operatorname{Proj} k[x, y, z, t]/(x^3 + y^3 + c_0 z^3 + d_0 t^3)$ for $P = (c_0, d_0) \in k^* \times k^*$.
- $e_1 = \left\{ \dfrac{d}{c}, \dfrac{x + \zeta y}{x + y} \right\}_3 \in \operatorname{Br}(X_F)$.

Theorem (Theorem 1, U.)

$\forall P = (c_0, d_0) \in k^* \times k^*$ *such that* c_0, d_0, $c_0 d_0$ *and* $d_0/c_0 \notin (k^*)^3$, $\operatorname{sp}(e_1; P)$ *is a generator of the group* $\operatorname{Br}(X_P)/\operatorname{Br}(k) \cong \mathbb{Z}/3\,\mathbb{Z}$.

$x^3 + by^3 + cz^3 + dt^3 = 0$ case

- $\mathcal{O}_F = k[b, c, d]$, $F = k(b, c, d)$.
- $X_F = \mathrm{Proj}\, F[x, y, z, t]/(x^3 + by^3 + cz^3 + dt^3)$.
- $X_P = \mathrm{Proj}\, k[x, y, z, t]/(x^3 + b_0 y^3 + c_0 z^3 + d_0 t^3)$ for $P = (b_0, c_0, d_0) \in k^* \times k^* \times k^*$.

Definition ($\mathcal{P}_k$)

$$\mathcal{P}_k := \{P \in (k^*)^{\times 3} \mid \mathrm{Br}(X_P)/\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}\}.$$

Proposition

The following conditions are equivalent:

- $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$.
- $\mathcal{P}_k$ *is Zariski dense in* $(\mathbb{G}_m)^3$.

$x^3 + by^3 + cz^3 + dt^3 = 0$ case

- $\mathcal{O}_F$, F, X_F, X_P: as above.

Theorem (Theorem2, U.)

Assume $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$. *Then there is no element* $e \in \mathrm{Br}(X_F)$ *satisfying the following condition:*

> *there exists a dense open subset* $W \subset (\mathbb{G}_{m,k})^3$ *such that*
>
> - $\mathrm{sp}(e; \cdot)$ *is defined on* $W(k) \cap \mathcal{P}_k$*;*
> - *for all* $P \in W(k) \cap \mathcal{P}_k$, $\mathrm{sp}(e; P)$ *is a generator of the group* $\mathrm{Br}(X_P)/\mathrm{Br}(k)$.

An exact sequence

Let X be a smooth projective diagonal cubic surface over k defined by

$$x^3 + y^3 + cz^3 + dt^3 = 0.$$

We start the following exact sequence:

A fundamental exact sequence

$$0 \longrightarrow \mathrm{Br}_1(X)/\mathrm{Br}(k) \longrightarrow H^1(k, \mathrm{Pic}(\overline{X})) \stackrel{d^{1,1}}{\longrightarrow} H^3(k, \overline{k}^*),$$

where

- $\overline{X} = X \times_k \overline{k}$.
- $\mathrm{Pic}(\overline{X})$: Picard group of $\overline{X}$.
- $\mathrm{Br}_1(X) := \mathrm{Ker}(\mathrm{Br}(X) \longrightarrow \mathrm{Br}(\overline{X}))$.

Some reductions

Observation

For our X, we have

- $X(k) \neq \emptyset$
  $\Rightarrow d^{1,1} = 0$
  $\Rightarrow \mathrm{Br}_1(X)/\mathrm{Br}(k) \cong H^1(k, \mathrm{Pic}(\overline{X}))$.
- X is (geometrically) rational
  $\Rightarrow \mathrm{Br}(\overline{X}) = 0$
  $\Rightarrow \mathrm{Br}_1(X) = \mathrm{Br}(X)$.

$$\therefore \mathrm{Br}(X)/\mathrm{Br}(k) \cong H^1(k, \mathrm{Pic}(\overline{X}))$$

$\Rightarrow$ We can describe the Brauer group in terms of divisors!

Divisors on X

- $\gamma := (d/c)^{1/3}$, $k' = k(\gamma)$, $G = \mathrm{Gal}(k'/k)$.
- For $i = 0, 1, 2$, consider the following lines on X:

$$L(i) : x + \zeta^i y = z + \zeta^i \gamma t = 0,$$
$$L'(i) : x + \zeta^i y = z + \zeta^{i+1} \gamma t = 0,$$
$$L''(i) : x + \zeta^i y = z + \zeta^{i+2} \gamma t = 0.$$

- H: hyperplane section defined by $\{x = 0\}$.
- $\mathcal{D} := \mathbb{Z}\, H \oplus \bigoplus_{i=0}^{2} \mathbb{Z}\, L(i) \oplus \bigoplus_{i=0}^{2} \mathbb{Z}\, L'(i) \oplus \bigoplus_{i=0}^{2} \mathbb{Z}\, L''(i)$.
- $\mathcal{D}_0 := \mathrm{Ker}(\mathcal{D} \to \mathrm{Pic}(X_{k'}))$.

Divisors on X

Proposition

- *We have an exact sequence:*

$$0 \to \mathcal{D}_0 \to \mathcal{D} \to \mathrm{Pic}(X_{k'}) \to 0$$

- *We have the following isomorphisms:*

$$\begin{aligned} H^1(k, \mathrm{Pic}(\overline{X})) &\cong H^1(G, \mathrm{Pic}(X_{k'})) \\ &\cong \frac{\mathcal{D}_0 \cap N_G \mathcal{D}}{N_G \mathcal{D}_0} \\ &\cong \mathbb{Z}/3\mathbb{Z}. \end{aligned}$$

An commutative diagram

Put $\mathrm{Br}(X_{k'}/X) := \mathrm{Ker}(\mathrm{Br}(X) \to \mathrm{Br}(X_{k'}))$.
We have the following diagram with all rows and columns exact:

$$\begin{array}{ccccccc}
 & & \mathrm{Br}(k'/k) & = & \mathrm{Br}(k'/k) & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Br}(X_{k'}/X) & \longrightarrow & H^2(G, k'(X)^*) & \xrightarrow{\mathrm{div}} & H^2(G, \mathrm{Div}(X_{k'})) \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & H^1(G, \mathrm{Pic}(X_{k'})) & \xrightarrow{\partial'} & H^2(G, k'(X)^*/k'^*) & \xrightarrow{\mathrm{div}} & H^2(G, \mathrm{Div}(X_{k'}))
\end{array}$$

By chasing this diagram, we get a symbolic generator of $\mathrm{Br}(X)/\mathrm{Br}(k)$. □

Recall the statement of Theorem 2.

- $\mathcal{O}_F = k[b, c, d]$, $F = k(b, c, d)$.
- $X_F = \operatorname{Proj} F[x, y, z, t]/(x^3 + by^3 + cz^3 + dt^3)$.
- $X_P = \operatorname{Proj} k[x, y, z, t]/(x^3 + b_0y^3 + c_0z^3 + d_0t^3)$ for $P = (b_0, c_0, d_0) \in k^* \times k^* \times k^*$.

Theorem (Theorem 2, U.)

Assume $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$. *Then there is no element* $e \in \operatorname{Br}(X_F)$ *satisfying the following condition:*

there exists a dense open subset $W \subset (\mathbb{G}_{m,k})^3$ *such that*

- $\operatorname{sp}(e; \cdot)$ *is defined on* $W(k) \cap \mathcal{P}_k$;
- *for all* $P \in W(k) \cap \mathcal{P}_k$, $\operatorname{sp}(e; P)$ *is a generator of the group* $\operatorname{Br}(X_P)/\operatorname{Br}(k)$.

A vanishing theorem

- $F = k(\lambda, \mu, \nu)$.
- $X = \operatorname{Proj} F[x, y, z, t]/(x^3 + \lambda y^3 + \mu z^3 + \lambda\mu\nu t^3)$.

Theorem 2 is a consequence of the following theorem:

Theorem

$$\operatorname{Br}(X)/\operatorname{Br}(F) = 0.$$

So we are devoted to the proof of this theorem in the following argument.

- $\alpha = \lambda^{1/3}$, $\alpha' = \mu^{1/3}$, $\gamma = \nu^{1/3}$.
- $F' = F(\alpha, \gamma)$, $F'' = F'(\alpha')$.

Goal and Strategy

Fact

We have an exact sequence:

$$0 \to \mathrm{Br}(X)/\mathrm{Br}(F) \to H^1(F, \mathrm{Pic}(\overline{X})) \xrightarrow{d^{1,1}} H^3(F, \overline{F}^*)$$

and isomorphisms:

$$H^1(F, \mathrm{Pic}(\overline{X})) \cong H^1(F'/F, \mathrm{Pic}(X_{F'})) \cong \mathbb{Z}/3\mathbb{Z}.$$

Our goal

To prove:

a generator $\phi \in H^1(F'/F, \mathrm{Pic}(X_{F'}))$ *does not vanish in* $H^3(F, \overline{F}^*)$.

In the following, we may assume k contains all roots of unity.

Goal and Strategy

Problem

$H^3(F, \overline{F}^*)$ is too big to check ϕ is zero or not in it...

$\Rightarrow$So we have to choose a smaller (finite) group G so that the image of ϕ in G is nonzero.

Strategy

- Step 1: Explicit description of $d^{1,1}$.
- Step 2: Consider $\phi \in H^3(F, \overline{F}^*)$ as an element of $H^3(F''/F, \mu_3)$.
- Step 3: Reduction to H^2.
- Step 4: Reduction to H^1.

Step 1: Explicit description of $d^{1,1}$

- $0 \to \mathcal{D}_0 \to \mathcal{D} \to \mathrm{Pic}(X_{F'}) \to 0$
- $0 \to F'^* \to \mathrm{div}^{-1}(\mathcal{D}_0) \to \mathcal{D}_0 \to 0$

Proposition (Kresch-Tschinkel, 2006.)

We have the following commutative diagram.

$$\begin{array}{ccc} H^1(F'/F, \mathrm{Pic}(X_{F'})) & \xrightarrow{\partial} & H^2(F'/F, \mathcal{D}_0) \\ & \searrow^{d^{1,1}_{F'}} & \downarrow \delta \\ & & H^3(F'/F, F'^*). \end{array}$$

Using this result, we get an explicit cocycle $\phi \in H^3(F'/F, (F')^*)$.

Step 2: ϕ as an element of $H^3(F''/F, \mu_3)$

Using the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
 & & H^3(F'/F, F'^*) & & \\
 & & \downarrow i^{F'}_{F''} & & \\
H^3(F''/F, \mu_3) & \longrightarrow & H^3(F''/F, F''^*) & \xrightarrow{3} & H^3(F''/F, (F''^*)^3) \\
\downarrow & & \downarrow i^{F''}_{F} & & \downarrow \\
H^3(F, \mu_3) & \longrightarrow & H^3(F, \overline{F}^*) & \xrightarrow{3} & H^3(F, \overline{F}^*),
\end{array}$$

We can consider ϕ as an element of $H^3(F''/F, \mu_3)$.

Step 3: Reduction to H^2

- $D \subset \mathbb{A}^3_k$: a principal divisor, $k(D)$: its residue field.
- For each D, we have a comm. diagram:

$$
\begin{array}{ccccc}
H^3(F''/F, \mu_3) & & & & \\
\big\downarrow {\scriptstyle i^{F''}_{F}} & & & & \\
H^3(F, \mu_3) & \xrightarrow{\mathrm{res}_D} & H^2(k(D), \mathbb{Z}/3\mathbb{Z}) & \xrightarrow{\cong} & H^2(k(D), \mu_3) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
H^3(F, \mathbb{Q}/\mathbb{Z}(1)) & \xrightarrow{\mathrm{res}_D} & H^2(k(D), \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\cong} & H^2(k(D), \mathbb{Q}/\mathbb{Z}(1)) \\
\cong \big\downarrow & & & & \cong \big\downarrow \\
H^3(F, \overline{F}^*) & & & & H^2(k(D), \overline{k(D)}^*).
\end{array}
$$

Hence it suffices to prove the non-vanishingness of ϕ in $H^2(k(D), \mu_3)$ for some D.

Step 3: Reduction to H^2

- Take $D = \{\mu = 0\} \cong \mathbb{A}^2_k$.
- $k(D) = k(\lambda, \nu)$.
- $k(D)' := k(D)(\alpha), \quad \alpha = \lambda^{1/3}$.
- We have the inflation map

$$\inf\colon H^2(k(D)'/k(D), \mu_3) \to H^2(k(D), \mu_3).$$

Lemma (Reduction to finite group)

There exists $r\phi \in H^2(k(D)'/k(D), \mu_3)$ *such that*

$$\mathrm{res}_D(\phi) = \inf(r\phi).$$

This cocycle $r\phi$ is used in Step 4.

Step 4: Reduction to H^1

- For simplicity, put

$$E := k(D), \quad E' := k(D)', \quad \Phi = \mathrm{res}_D(\phi).$$

- Put $D' := \{\nu = 0\} \subset \mathbb{A}^2_k = \mathrm{Spec}\, k[\lambda, \nu]$.
- $k(D')(\cong k(\lambda))$: the residue field of D'.
- The residue map along to D':

$$\mathrm{res}_{D'} : H^2(E, \mu_3) \to H^1(k(D'), \mathbb{Z}/3\mathbb{Z}).$$

We consider the residue of $\Phi \in H^2(E, \mu_3)$ along D'.

Step 4: Reduction to H^1

Lemma

- $\mathrm{res}_{D'}(\Phi) \in H^1(k(D'), \mathbb{Z}/3\mathbb{Z})$ *comes from* $H^1(k(D')'/k(D'), \mathbb{Z}/3\mathbb{Z})$.
- *the inflation* $H^1(k(D')'/k(D'), \mathbb{Z}/3\mathbb{Z}) \to H^1(k(D'), \mathbb{Z}/3\mathbb{Z})$ *is injective.*

It's easy to check the nontriviality of $\phi \in H^1(k(D')'/k(D'), \mathbb{Z}/3\mathbb{Z})$. Therefore this complete the proof of Theorem 2. □

For more detail, see

http://www.ms.u-tokyo.ac.jp/˜tetsuya1/en/paper.html.

Thank you for your attention!
However, we have a few minutes and I want to talk an additional topic...
As an application, we can compute Chow groups of some diagonal cubic surfaces, by using symbolic generators.

Algebraic cycles

- X: a variety over a field k.
- $X_{(r)}$: the set of irr. cl. subsch. of X with dimension r.
- $Z_r(X) = \bigoplus_{D \in X_{(r)}} \mathbb{Z}[D]$: the group of r-cycles on X.
- $Z_r(X)_{rat} = \langle \mathrm{div}(f) \mid f \in k(W)^*, \exists W \in X_{(r+1)} \rangle_{\mathbb{Z}}$

Definition (Chow group)

$$\mathrm{CH}_r(X) := Z_r(X)/Z_r(X)_{rat}$$

Definition (degree map)

$$\deg\colon\ \mathrm{CH}_0(X) \to \mathbb{Z}; \sum n_P[P] \mapsto \sum n_P[k(P) : k].$$

Put $A_0(X) := \mathrm{Ker}(\deg\colon\ \mathrm{CH}_0(X) \to \mathbb{Z})$.

Brauer-Manin pairing

- k: p-adic field (fin. ext. of $\mathbb{Q}_p$), $\mathbb{F}$: its residue field.
- $\mathrm{inv}_k : \mathrm{Br}(k) \longrightarrow \mathbb{Q}/\mathbb{Z}$: invariant map of k.
- X: a variety over k.

Definition (Brauer-Manin pairing)

$$\langle \cdot, \cdot \rangle : \mathrm{CH}_0(X) \times \mathrm{Br}(X) \rightarrow \mathbb{Q}/\mathbb{Z};$$
$$\langle \sum n_P[P], \mathcal{A} \rangle = \sum n_P \,\mathrm{inv}_k(\mathrm{cores}_{k(P)/k}(P^* \mathcal{A})).$$

This pairing induces a map:

$$\phi_X : A_0(X) \rightarrow \mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(k), \mathbb{Q}/\mathbb{Z}).$$

Known results of ϕ_X

Theorem (Colliot-Thélène, 1983.)

If X is a rational surface, then the map ϕ_X is injective.

Theorem (Saito-Sato, 2009.)

- $\mathcal{U}$ *is regular, faithfully flat over* $\mathcal{O}_k$.
- $X := \mathcal{U} \times_{\mathcal{O}_k} k$*: smooth over* k, $Y := \mathcal{U} \times_{\mathcal{O}_k} \mathbb{F}$
- η*: a generic point of* Y
- $A_\eta := \mathcal{O}^h_{\mathcal{U},\eta}$, K_η*: the fractional field of* A_η.
- $\iota : \mathrm{Br}(X) \to \mathrm{Br}(K_\eta)$.

Assume $\iota^{-1}(\mathrm{Br}(A_\eta)) = 0$. *Then* ϕ_X *is surjective.*

If we can confirm the condition $\iota^{-1}(\mathrm{Br}(A_\eta)) = 0$ by using the symbolic representation of $\mathrm{Br}(X)$, we decide the structure of $A_0(X)$.

- k: p-adic field, ord_k: the normalized valuation.

Theorem (Saito-Sato, 2009.)

Let $X = \mathrm{Proj}\, k[x, y, z, t]/(x^3 + y^3 + z^3 + dt^3)$.

1. *Assume* $p \neq 3$. *Then*

$$A_0(X) = \begin{cases} 0 & \text{if } \mathrm{ord}_k(d) \equiv 0(3), \\ \mathbb{Z}/3\mathbb{Z} & \text{if } \mathrm{ord}_k(d) \not\equiv 0 \text{ and } \mu_3 \not\subset k, \\ (\mathbb{Z}/3\mathbb{Z})^2 & \text{if } \mathrm{ord}_k(d) \not\equiv 0 \text{ and } \mu_3 \subset k. \end{cases}$$

2. *Assume* $p = 3$, $\mu_3 \subset k$ *and* $\mathrm{ord}(d) \equiv 1(3)$. *Then*

$$A_0(X) = (\mathbb{Z}/3\mathbb{Z})^2.$$

Theorem (U.)

Let $X = \mathbf{Proj}\, k[x,y,z,t]/(x^3+y^3+cz^3+dt^3)$ *with* c, d, cd *and* $d/c \notin (k^*)^3$.

1. *Assume* $p \neq 3$. *Then*

$$A_0(X) = \begin{cases} 0 & \text{if } \mathrm{ord}_k(c) \equiv \mathrm{ord}_k(d) \equiv 0(3), \\ \mathbb{Z}/3\mathbb{Z} & \text{otherwise.} \end{cases}$$

2. *Assume* $p = 3$, $\mu_3 \subset k$ *and*

$$\mathrm{ord}_k(c) = 0, \mathrm{ord}_k((c-1)/3) > 0, \mathrm{ord}_k(d) \equiv 1(3).$$

Then

$$A_0(X) = \mathbb{Z}/3\mathbb{Z}.$$

Thank you for your attention!