CONTINUITY OF THE LOCAL EVALUATION MAPS

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ABSTRACT. We show the continuity of the local evaluation map naturally induced by the Brauer-Manin pairing.

1. INTRODUCTION

In the following, $\mathrm{Br}(X)$ always denotes the cohomological Brauer group $H^2(X, \mathbb{G}_m)$ of a scheme X. Let k be a number field and X be a proper smooth variety over k. Manin constructed what is now called the Brauer-Manin pairing:

$$\langle \cdot, \cdot \rangle \colon \prod_v X(k_v) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}; \quad \langle (x_v), \mathcal{A} \rangle = \sum_v \mathrm{inv}_v \, \mathcal{A}(x_v).$$

To analyse this pairing, it is of course important to study each local pairing:

$$X(k_v) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}.$$

If we fix an element $\mathcal{A} \in \mathrm{Br}(X)$, we have the map $\phi_{\mathcal{A}} \colon X(k_v) \to \mathbb{Q}/\mathbb{Z}$ and call it the local evaluation map. We naturally want to know the property of this map. Here, we have the following continuity theorem:

Theorem 1.1. *Let k be a local field, X be a smooth scheme over k, and $\mathcal{A} \in \mathrm{Br}(X)$. Then the local evaluation map $\phi_{\mathcal{A}} \colon X(k) \to \mathbb{Q}/\mathbb{Z}$ is locally constant. Here, the set $X(k)$ is considered as an analytic topological space.*

2. PROOF OF THE THEOREM

To prove Theorem 1.1, we prove the following result:

Theorem 2.1. *Let k be a local field, G a algebraic group over k, X a smooth variety over k and $f \colon Y \to X$ be a right X-torsor under G. Then the following map*

$$X(k) \to \check{H}^1(k, G); P \mapsto [Y_P]$$

is locally constant.

Before proving this theorem, we first show that this theorem implies the main theorem:

Theorem 2.1 ⇒ Theorem 1.1. For each positive integer n, we have the map

$$\delta_n : H^1(X, PGL_n) \to H^2(X, \mathbb{G}_m) = \mathrm{Br}(X)$$

and the equality

$$\bigcup_n \mathrm{Im}(\delta_n) = \mathrm{Br}(X).$$

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Therefore we can take a right X-torsor Y under PGL_n such that the image of its class $[Y] \in H^1(X, PGL_n)$ under δ_n is $\mathcal{A}$. We now apply Theorem 2.1 to $G = PGL_n$ and get the locally constant map

$$X(k) \to H^1(k, PGL_n) \to \mathrm{Br}(k).$$

This map is no other than the local evaluation map $\phi_{\mathcal{A}}$, which proves Theorem 1.1. ☐

Proof of Theorem 2.1. First we reduce to the case $[Y_P] = 0$. If $[Y_P] \neq 0$, we replace Y with the contracted product $Y \times^G (Y_P)_X$, i.e. the quotient $Y \times_X (Y_P)_X / G_X$. Here Y_P is considered as a left k-torsor under G and the G_X-action on $Y \times_X (Y_P)_X$ is defined to be

$$g \cdot (p, q) = (pg^{-1}, gq).$$

$Y \times^G (Y_P)_X$ is a right X-torsor under $G \times^G (Y_P)_X \cong (G \times^G Y_P) \times_k X$. For details, see [Sk01].

For each $Q \in X(k)$, we have the following commutative diagram in the category of sets:

$$\begin{tikzcd}
H^1(X, G) \arrow[r, "\cdot \times^G (Y_P)_X", "\cong"'] \arrow[d, "Q^*"'] & H^1(X, G \times^G (Y_P)_X) \arrow[d, "Q^*"] \\
H^1(k, G) \arrow[r, "\cong", "\cdot \times^G Y_P"'] & H^1(k, G \times^G Y_P).
\end{tikzcd}$$

If $Q = P$, the class $[Y_P] \in H^1(k, G)$ corresponds to the class $[Y_P \times^G Y_P] = 0$. Hence we may assume $[Y_P] = 0$ in advance.

Since $[Y_P] = 0$, we have a section, i.e. a k-rational point in $Y_P(k)$. Then we have the following lemma:

Lemma 2.2. *Let k be a local field, X, Y be smooth k-varieties and $f : Y \to X$ be a morphism. $f(k) : Y(k) \to X(k)$ denotes the corresponding morphism between k-analytic manifolds $Y(k)$ and $X(k)$. Assume f is étale at a k-rational point $P \in Y$. Then $f(k)$ is also étale at $P \in Y(k)$ in the analytic sense.*

Proof. Since f is étale at P, we have the following isomorphism of cotangent spaces:

$$\mathfrak{m}_{Y,P}/\mathfrak{m}_{Y,P}^2 \cong \mathfrak{m}_{X,Q}/\mathfrak{m}_{X,Q}^2,$$

where $Q = f(P)$. Using the isomorphisms

$$\hat{\mathcal{O}}_{Y,P} \cong \hat{\mathcal{O}}^{an}_{Y,P}, \quad \hat{\mathcal{O}}_{X,Q} \cong \hat{\mathcal{O}}^{an}_{X,Q},$$

this implies the isomorphism of analytic cotangent spaces:

$$\mathfrak{m}^{an}_{Y,P}/{\mathfrak{m}^{an}_{Y,P}}^2 \cong \mathfrak{m}^{an}_{X,Q}/{\mathfrak{m}^{an}_{X,Q}}^2,$$

which says that $f(k)$ is étale at P. ☐

Using this lemma, we can take a open neighborhood $U \subset X(k)$ at P and k-section $s : U \to Y(k)$ of $f(k)$. Therefore, for any $P' \in U$, the fiber $Y_{P'}$ has a rational point $s(P')$, in particular, we have $[Y_{P'}] = 0$ in $H^1(k, G)$. This proves the desired local constancy. ☐

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