

# BRAUER GROUP OF PROJECTIVE SPACES

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ABSTRACT. We show a well-known result which states that the Brauer group of a projective space over a field is isomorphic to that of the base field.

## 1. INTRODUCTION

In the following,  $\mathrm{Br}(X)$  always denotes the cohomological Brauer group  $H^2(X, \mathbb{G}_m)$  of a scheme  $X$ . We prove the following theorem:

**Theorem 1.1.** *Let  $k$  be a field of characteristic zero,  $n$  a positive natural number and  $\pi: \mathbb{P}_k^n \rightarrow \mathrm{Spec} k$  be the structure morphism of the  $n$ -dimensional projective space over  $k$ . Then we have the isomorphism*

$$\pi^*: \mathrm{Br}(k) \xrightarrow{\cong} \mathrm{Br}(\mathbb{P}_k^n).$$

I do not know clearly that this result holds for the positive characteristic case. Our proof, which reduces an argument for  $\mathbb{G}_m$  to that of torsion sheaves  $\mu_m$ , cannot be applied to the positive characteristic case.

We have an immediate corollary:

**Corollary 1.2.** *Let  $k$  be an algebraically closed field of characteristic zero. Then we have  $\mathrm{Br}(\mathbb{P}_k^n) = 0$ .*

Using the birational invariance of Brauer groups, due to Grothendieck [G68], we have the following corollary:

**Corollary 1.3.** *Let  $k$  be a field of characteristic zero and  $X$  be a proper rational variety over  $k$ . Then  $\mathrm{Br}(X) \cong \mathrm{Br}(k)$ .*

## 2. PROOF OF THE THEOREM

We first note that the map  $\pi^*: \mathrm{Br}(k) \rightarrow \mathrm{Br}(\mathbb{P}_k^n)$  is injective, since  $\mathbb{P}_k^n$  has a rational point. Hence we are devoted to prove the surjectivity of  $\pi^*$ .

**Lemma 2.1.**  *$\mathrm{Br}(\mathbb{P}_k^n)$  is torsion.*

*Proof.* Since  $\mathbb{P}_k^n$  is a smooth integral variety,  $\mathrm{Br}(\mathbb{P}_k^n)$  injects the Brauer group  $\mathrm{Br}(k(\mathbb{P}_k^n))$  of the function field of  $\mathbb{P}_k^n$ . For a proof, see [G68]. Moreover, the Brauer group of a field is isomorphic to Galois cohomology group, which is torsion.  $\square$

By this Lemma, in order to prove the theorem, it suffices to show  ${}_m\mathrm{Br}(k) \cong {}_m\mathrm{Br}(\mathbb{P}_k^n)$  for all  $k \in \mathbb{Z}$ .

**Lemma 2.2.** *We have the following exact sequence of groups:*

$$\mathrm{Pic}(\mathbb{P}_k^n) \xrightarrow{d} H^2(\mathbb{P}_k^n, \mu_m) \rightarrow {}_m\mathrm{Br}(\mathbb{P}_k^n) \rightarrow 0.$$

*Proof.* We have the Kummer sequence of étale sheaves on  $\mathbb{P}_k^n$ :

$$1 \rightarrow \mu_m \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 1.$$

Taking the cohomology long exact sequence, we easily obtain the desired exact sequence.  $\square$

**Lemma 2.3.** *We have the following exact sequence of groups:*

$$0 \rightarrow {}_m\mathrm{Br}(k) \rightarrow H^2(\mathbb{P}_k^n, \mu_m) \rightarrow H^2(\mathbb{P}_k^n, \mu_m)^{G_k} \rightarrow 0.$$

*Proof.* We use the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(k, H^q(\mathbb{P}_{\bar{k}}^n, \mu_m)) \Rightarrow H^{p+q}(\mathbb{P}_{\bar{k}}^n, \mu_m).$$

Noting that  $E_2^{p,1} = 0$  for all  $p$  since we have

$$H^1(\mathbb{P}_{\bar{k}}^n, \mu_m) \cong \bar{k}^* / (\bar{k}^*)^m = 0,$$

we find  $E_2^{0,2} \cong E_\infty^{0,2}$  and the following exact sequence

$$0 \rightarrow E_\infty^{2,0} \rightarrow E^2 \rightarrow E_\infty^{2,0} \rightarrow 0,$$

from which the claim easily follows.  $\square$

**Lemma 2.4.**  $H^2(\mathbb{P}_{\bar{k}}^n, \mu_m) \cong \mathbb{Z}/m\mathbb{Z}$ .

*Proof.* In the proof, we always consider over  $\bar{k}$ . Let  $X = \mathbb{P}^n$ ,  $Z = \mathbb{P}^{n-1}$ , which is considered as a closed subscheme  $i: Z \rightarrow X$ . Then the complement  $U = X \setminus Z$  is isomorphic to  $\mathbb{A}^n$ . Since the pair  $(Z, X)$  is a smooth pair with codimension 1, this satisfies the purity theorem:

$$R^j i^! \mu_m = \begin{cases} 0 & j \neq 2 \\ \mathbb{Z}/m\mathbb{Z} & j = 2. \end{cases}$$

In particular, we have the following Gysin sequence:

$$\cdots \rightarrow H^1(U, \mu_m) \rightarrow H^0(Z, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{i_*} H^2(X, \mu_m) \rightarrow H^2(U, \mu_m) \rightarrow \cdots.$$

Since  $g: U = \mathbb{A}^n \rightarrow \text{Spec } \bar{k}$  is an acyclic morphism, that is,  $\mathcal{F} \cong g_* g^* \mathcal{F}$  and  $R^j g_* g^* \mathcal{F} = 0$  for all torsion sheaves  $\mathcal{F}$  and all integers  $i > 0$ , we have

$$\forall j > 0, \quad H^j(U, \mu_m) = 0.$$

Therefore we get

$$H^2(X, \mu_m) \cong H^0(Z, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z},$$

which completes the proof of this lemma.  $\square$

**Lemma 2.5.** *the map  $d: \text{Pic}(\mathbb{P}_{\bar{k}}^n) \rightarrow H^2(\mathbb{P}_{\bar{k}}^n, \mu_m)$  is surjective.*

*Proof.* We know  $\text{Pic}(\mathbb{P}_{\bar{k}}^n) \cong \mathbb{Z}$  and  $H^2(\mathbb{P}_{\bar{k}}^n, \mu_m) \cong \mathbb{Z}/m\mathbb{Z}$ . Putting these into the following exact sequence

$$\cdots \rightarrow \text{Pic}(\mathbb{P}_{\bar{k}}^n) \xrightarrow{m} \text{Pic}(\mathbb{P}_{\bar{k}}^n) \rightarrow H^2(\mathbb{P}_{\bar{k}}^n, \mu_m) \rightarrow \cdots,$$

we can immediately see that  $d$  is surjective.  $\square$

*Proof of the theorem.* Using the above lemmas, we get the following commutative diagram:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \uparrow & & \\ & & & & {}_m\text{Br}(\mathbb{P}_{\bar{k}}^n) & & \\ & & & \nearrow \pi^* & \uparrow & & \\ 0 & \longrightarrow & {}_m\text{Br}(k) & \longrightarrow & H^2(\mathbb{P}_{\bar{k}}^n, \mu_m) & \longrightarrow & H^2(\mathbb{P}_{\bar{k}}^n, \mu_m)^{G_k} \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & & & \text{Pic}(\mathbb{P}_{\bar{k}}^n) & \xrightarrow{\cong} & \text{Pic}(\mathbb{P}_{\bar{k}}^n). \end{array}$$

By an usual diagram chasing, we can prove the surjectivity of the map

$$\pi^*: {}_m\text{Br}(k) \rightarrow {}_m\text{Br}(\mathbb{P}_{\bar{k}}^n),$$

which completes the proof of the theorem.  $\square$

## REFERENCES

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