AN EXACT SEQUENCE ATTACHED TO BRAUER GROUPS

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Abstract. We give a useful exact sequence attached to Brauer groups of schemes. This relates Brauer group with some unramified cohomology.

1. Introduction

In the following, $\mathrm{Br}(X)$ always denotes the cohomological Brauer group $H^2(X, \mathbb{G}_m)$ of a scheme X. $X^{(i)}$ denotes the set of points in X with codimension i.

We prove the following theorem:

Theorem 1.1. *Let X be a regular integral scheme, $k(X)$ be the function field of X and $\mathbb{P}$ be the set of primes $\{p \mid \exists x \in X \ \ \mathrm{ch}(k(x)) = p\}$. Then for each prime $\ell \notin \mathbb{P}$, we have the following exact sequence:*

$$0 \to \mathrm{Br}(X)\{\ell\} \to \mathrm{Br}(k(X))\{\ell\} \to \bigoplus_{x \in X^{(1)}} H^1(k(x), \mathbb{Q}/\mathbb{Z})\{\ell\}.$$

Here, for a group A, $A\{\ell\}$ denotes the ℓ-primary subgroup of A.

2. Proof of the theorem

We begin with the following lemma:

Lemma 2.1. *Let X be a regular integral scheme. Then we have the canonical inclusion*

$$\mathrm{Br}(X) \to \mathrm{Br}(k(X)).$$

Proof. By the regularity of X, we have the following exact sequence:

$$0 \to \mathbb{G}_m \to j_* \mathbb{G}_{m,\eta} \to \bigoplus_{x \in X^{(1)}} i_{x*} \mathbb{Z} \to 0, \tag{1}$$

where $j : \eta = \operatorname{Spec} k(X) \hookrightarrow X$ and $i_x : x = \operatorname{Spec} k(x) \hookrightarrow X$. Thus we get the exact sequence

$$\cdots \to \bigoplus_{x \in X^{(1)}} H^1(X, i_{x*} \mathbb{Z}) \to H^2(X, \mathbb{G}_m) \to H^2(X, j_* \mathbb{G}_{m,\eta}) \to \cdots.$$

We can also prove $H^1(X, i_{x*} \mathbb{Z}) = 0$ for all $x \in X^{(1)}$ and $H^2(X, j_* \mathbb{G}_{m,\eta}) \cong H^2(k(X), \mathbb{G}_{m,\eta})$, which completes the proof of Lemma 2.1. See also [G68], II, Corollaire 1.8. □

Lemma 2.2 (Mayer-Vietoris exact sequence)**.** *Let X be a scheme, $\{U, V\}$ be a Zariski open covering of X and $\mathcal{F}$ be a étale sheaf on X. Then we have the following exact sequence:*

$$\cdots \to H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \oplus H^i(V, \mathcal{F}) \to H^i(U \cap V, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}) \to \cdots.$$

Lemma 2.3 (Absolute purity theorem)**.** *Let $i\colon Y \hookrightarrow X$ be a closed immersion of noetherian regular schemes of pure codimension c. Let n be an integer which is invertible on X. Then*

$$\mathcal{H}^q_Y(\mu_n^{\otimes j}) \cong \begin{cases} 0 & q \neq 2c \\ \mu_n^{\otimes j-c} & q = 2c. \end{cases}$$

Proof. See [F02]. □

Lemma 2.4 (Corllary of Lemma 2.3(Semi-purity theorem))**.** *Let X be a noetherian regular scheme and n be an integer which is invertible on X. Then for (not necessarily regular) closed subscheme Y of pure codimension $\geq c$,*

$$\mathcal{H}^q_Y(\mu_n^{\otimes j}) = 0, \forall q < 2c.$$

Date: Sep. 19, 2012.

Lemma 2.5 (Corollary of Lemma 2.3)**.** *For a discrete valuation ring A with quotient field K and residue field κ and for an integer n coprime to $\operatorname{ch}\kappa$, we have the following exact sequence:*

$$H^2(A,\mu_n^{\otimes j}) \to H^2(K,\mu_n^{\otimes j}) \xrightarrow{\partial_A} H^1(\kappa,\mu_n^{\otimes j-1}).$$

Proof. Put $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec}\kappa$. Then $c = 1$ and the complement $U = \operatorname{Spec} K$. We have the following local-global spectral sequence:

$$H^p(Y,\mathcal{H}^q_Y(\mu_n^{\otimes j})) \Rightarrow H^{p+q}_Y(X,\mu_n^{\otimes j}).$$

Applying Theorem 2.3 to this spectral sequence, we get

$$H^p(Y,\mu_n^{\otimes j-1}) \cong H^{p+2}_Y(X,\mu_n^{\otimes j}).$$

On the othe hand, we have the localization sequence:

$$\cdots \to H^i_Y(X,\mu_n^{\otimes j}) \to H^i(X,\mu_n^{\otimes j}) \to H^i(U,\mu_n^{\otimes j}) \to H^{i+1}_Y(X,\mu_n^{\otimes j}) \to \cdots.$$

Conbining these results, we have the following exact sequence, which is called "Gysin sequence":

$$\cdots \to H^{i-2}(Y,\mu_n^{\otimes j-1}) \to H^i(X,\mu_n^{\otimes j}) \to H^i(U,\mu_n^{\otimes j}) \to H^{i-1}(Y,\mu_n^{\otimes j-1}) \to \cdots.$$

If we put $i = 2$, then the claim is obtained. □

Remark 2.6. If A is complete, this residue map ∂_A is equal (up to a sign) to the residue map defined by using the following spectral sequence

$$H^p(K^{\mathrm{ur}}/K, H^q(K^{\mathrm{ur}},\mu_n^{\otimes j})) \Rightarrow H^{p+q}(K,\mu_n^{\otimes j}).$$

Proposition 2.7. *For a noetherian regular integral scheme X and an integer n which is invertible on X, we have the following exact sequence:*

$$H^2(X,\mu_n^{\otimes j}) \to H^2(k(X),\mu_n^{\otimes j}) \xrightarrow{\bigoplus \partial_x} \bigoplus_{x\in X^{(1)}} H^1(k(x),\mu_n^{\otimes j-1}),$$

where $\partial_x = \partial_{\mathcal{O}_{X,x}}$ in Lemma 2.5

Proof. The above sequence is a complex since for each $x \in X^{(1)}$, we have the factorization

$$H^2(X,\mu_n^{\otimes j}) \to H^2(\mathcal{O}_{X,x},\mu_n^{\otimes j}) \to H^2(k(X),\mu_n^{\otimes j}) \to H^1(k(x),\mu_n^{\otimes j-1})$$

and the composite of the middle and right maps is the zero map by Lemma 2.5.

Take $\alpha \in \operatorname{Ker}(H^2(k(X),\mu_n^{\otimes j}) \to \bigoplus_{x\in X^{(1)}} H^1(k(x),\mu_n^{\otimes j-1})$. By the commutativity of étale cohomology and filtered direct limit, there exists an open $U \subset X$ and $\alpha_U \in H^2(U,\mu_n^{\otimes j})$ such that $\alpha_U \mapsto \alpha$. Put $Y := X\setminus U$ and $Y_1 \subset Y$ to be the union of irreducible components of Y with codimension 1 in X. Write Y_1 as $Y_1 = \bigcup_{i=1}^N D_i$, where D_i is a principal divisor. Since Y is noetherian, this union is indeed finite. Let x_i be the generic point of D_i. Since $\partial_{x_1}(\alpha) = 0$, there exists a $\alpha_{x_1} \in H^2(\mathcal{O}_{X,x_1},\mu_n^{\otimes j})$ such that $\alpha_{x_1} \mapsto \alpha$. Moreover, we can take a open neighborhood V_1 of x_1 and $\alpha_{V_1} \in H^2(V_1,\mu_n^{\otimes j})$ such that $\alpha_{V_1} \mapsto \alpha_{x_1}$. we would like to show α_U and α_{V_1}. Since $\varprojlim_{x_1\in V}(U\cap V) = U \cap \varprojlim_{x_1\in V} V = \operatorname{Spec} k(X)$, we can replace V_i with a smaller neighborhood of x_1 and assume that $\alpha_U|_{U\cap V_1} = \alpha_{V_1}|_{U\cap V_1}$. Hence we know there exists $\alpha_{U\cup V_1}$ which is a extension of α_U and α_{V_1}.

Applying the same argument to $V_2,\dots,V_N$, we obtain an element $\alpha_{U\cup V_1\cup\cdots\cup V_N}$ whose image in $H^2(k(X),\mu_n^{\otimes j})$ is α. Replace $U \cup V_1 \cup \cdots \cup V_N$ with U. Then the new $Y = X \setminus U$ has codimension at least 2 in X. Hence by semi-purity theorem(Lemma 2.4), we have

$$H^2(X,\mu_n^{\otimes j}) \cong H^2(X\setminus Y,\mu_n^{\otimes j}),$$

which means that we can get an element $\alpha_X \in H^2(X,\mu_n^{\otimes j})$ such that $\alpha_X \mapsto \alpha$. □

Proof of Theorem 1.1. In Proposition 2.7, put $n = \ell^m$ and $j = 1$. Then we have:

$$H^2(X,\mu_{\ell^m}) \to {}_{\ell^m}\mathrm{Br}(k(X)) \xrightarrow{\bigoplus \partial_x} H^1(k(x),\mathbb{Z}/\ell^m\mathbb{Z}).$$

On the other hand, by the Kummer sequence and Lemma 2.1, the left map above factors as follows:

$$H^2(X,\mu_{\ell^m}) \twoheadrightarrow {}_{\ell^m}\mathrm{Br}(X) \hookrightarrow {}_{\ell^m}\mathrm{Br}(k(X)).$$

Combining these sequence, we obtain the exact sequence:

$$0 \to {}_{\ell^m}\mathrm{Br}(X) \to {}_{\ell^m}\mathrm{Br}(k(X)) \xrightarrow{\bigoplus \partial_x} \bigoplus_{x\in X^{(1)}} H^1(k(x),\mathbb{Z}/\ell^m\mathbb{Z}).$$

Taking the direct limit with respect to m, we complete the proof of Theorem 1.1. □

References

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