$SL(2,\mathbb{C})$-*METABELIAN REPRESENTATIONS*

and ALGEBRAIC VARIETIES

$\sim$ *from* $SL(2,\mathbb{C})$*-representations to knot contact homology* $\sim$

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1. $SL(2,\mathbb{C})$-metabelian representations and their properties

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3. Abelian knot contact homology and $\mathcal{F}^{(2)}(K)$

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§1.0. Preliminaries

$G := \langle g_1, g_2 \cdots, g_r \mid r_i(g_1, \cdots, g_r) = 1 \ (i = 1, \cdots, s)\rangle$.

A representation $\rho : G \to SL(2, \mathbb{C})$ is said to be

(1) *reducible*: $\exists A \in SL(2, \mathbb{C})$ s.t. $A^{-1}\rho(g_i)A = \begin{bmatrix} \lambda_i & * \\ 0 & \lambda_i^{-1} \end{bmatrix}$

- *abelian* (**ex.** diagonal)

(2) *irreducible*: otherwise.

The *character of ρ*, denoted by χ_ρ, means the function

$$\chi_\rho(g) := \text{trace}(\rho(g)) =: t_g(\chi_\rho) \text{ or } t_g(\rho), \text{ for } g \in G.$$

Then the set of characters of all the representations

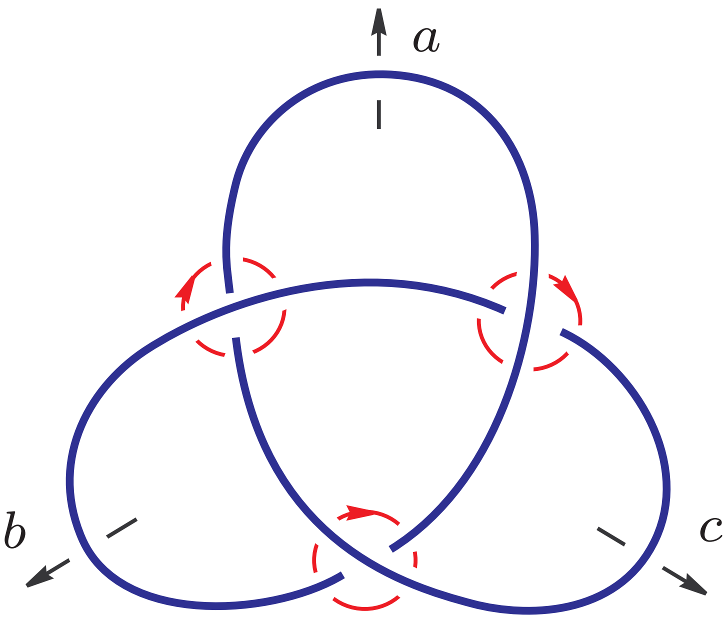
$$X(G) := \{\chi_\rho \mid \rho \in \text{Hom}(G, SL(2, \mathbb{C}))\}$$

can be identified with an algebraic variety in some complex space, so-called *$SL(2, \mathbb{C})$-character variety of G*.

Knot groups G_K

Let $E_K := S^3 - N(K)$. Then $G_K := \pi_1(E_K)$.

EX. $G_{3_1} = \langle a, b \mid abab^{-1}a^{-1}b^{-1} = 1 \rangle = \langle a, b, c \mid r_1 = 1, r_2 = 1, r_3 = 1 \rangle$.

$3_1 =$ 

Knot groups G_K

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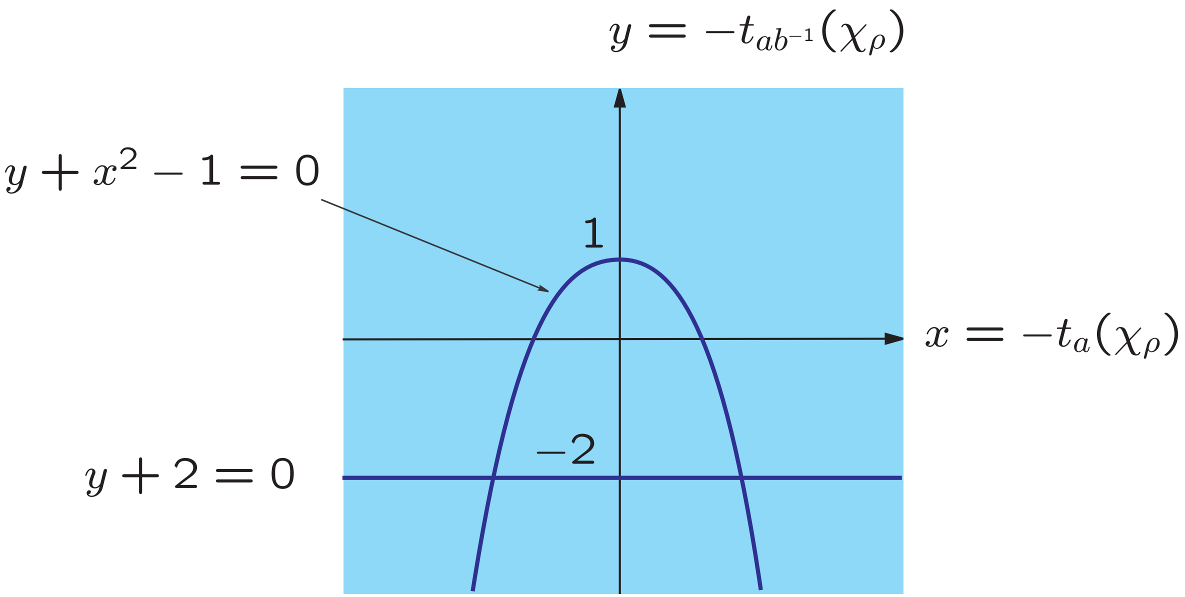
EX. $G_{3_1} = \langle a, b \mid abab^{-1}a^{-1}b^{-1} = 1 \rangle$.

$$X(G_{3_1}) = \{(x, y) \in \mathbb{C}^2 \mid (y+2)(y+x^2-1) = 0\}$$

where $x := -t_a(\chi_\rho)$, $y := -t_{ab^{-1}}(\chi_\rho)$

$X(G_{3_1})$ in $\mathbb{R}^2$ (genuine $X(G_{3_1})$ is in $\mathbb{C}^2$)



- $\mathrm{Hom}(G_{3_1}, SL(2,\mathbb{C})) \equiv \{(\rho(a),\rho(b)) \in SL(2,\mathbb{C})^2 \mid \rho(abab^{-1}a^{-1}b^{-1}) = \rho(1)\}$
$$= \left\{ (A,B) \in SL(2,\mathbb{C})^2 \,\middle|\, ABAB^{-1}A^{-1}B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
- $X(G_{3_1}) = \{\chi_\rho \mid \rho \in \mathrm{Hom}(G_{3_1}, SL(2,\mathbb{C}))\}$
$$\equiv \left\{ (\mathrm{trace}(A), \mathrm{trace}(AB^{-1})) \in \mathbb{C}^2 \,\middle|\, ABAB^{-1}A^{-1}B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\boxed{\chi_\rho \Leftrightarrow (\mathrm{trace}(A), \mathrm{trace}(AB^{-1}))}$$

$$\left(\begin{array}{c} \mathrm{trace}(X^{-1}) = \mathrm{trace}(X), \quad \mathrm{trace}(Y^{-1}XY) = \mathrm{trace}(X), \\ \mathrm{trace}(XY) = \mathrm{trace}(X)\mathrm{trace}(Y) - \mathrm{trace}(XY^{-1}), \quad X, Y \in SL(2,\mathbb{C}). \\ \mathrm{trace}(B) = \mathrm{trace}(A),\ \mathrm{trace}(A^{-1}) = \mathrm{trace}(A),\ \mathrm{trace}(BAB^{-1}) = \mathrm{trace}(A) \\ \mathrm{trace}(AB^{-1}A) = \mathrm{trace}(AB^{-1})\mathrm{trace}(A) - \mathrm{trace}(A), \cdots\cdots\cdots \end{array} \right)$$

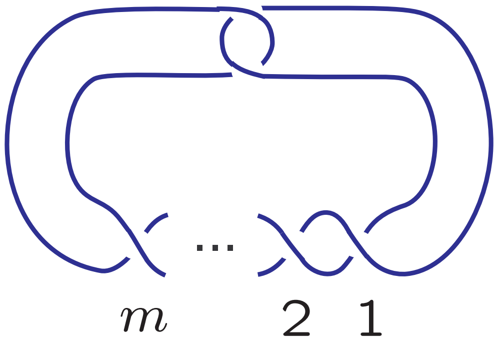
Put $w := \mathrm{trace}(A)$, $z := \mathrm{trace}(AB^{-1})$.

$$\boxed{X(G_{3_1}) = \left\{ (w,z) \in \mathbb{C}^2 \,\middle|\, (z-2)(z-w^2+1)^2 = 0 \right\}} \ \blacksquare$$

<u>Excecise.</u> Show that

$$\boxed{\mathrm{trace}(ABAB^{-1}A^{-1}B^{-1})} = z^3 - 2w^2z^2 + (w^4 + 2w^2 - 3)z - 2w^4 + 4w^2.$$

A more general fact: Let K_m be an m-twist knot

$K_m :=$  .

$(K_1 = 3_1,\ K_2 = 4_1,\ K_3 = 5_2,\ K_4 = 6_1,\ K_5 = 7_2,\ K_6 = 8_1,\ \cdots)$

[Gelca-N] & [N]

$$X(G_{K_m}) = \left\{(x,y) \in \mathbb{C}^2 \mid (y+2)\left(S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)\right) = 0\right\}$$

where $S_m(y)$ be *the Chebyshev polynomial* defined by

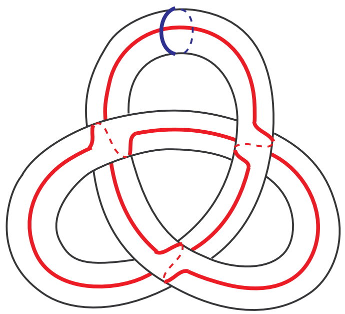
$$S_{m+2}(y) := yS_{m+1}(y) - S_m(y),\ \ S_1(y) := y,\ \ S_0(y) := 1.$$

§1.1. Metabelian representations and its characters

A representation $\rho : G \to SL(2,\mathbb{C})$ is said to be *metabelian*, if the commutator subgroup $[G,G] := \{xyx^{-1}y^{-1} \mid x,y \in G\}$ is mapped to an abelian subgroup in $SL(2,\mathbb{C})$.

(**Note:** abelian $\Rightarrow$ metabelian)

Let μ, λ be the elements in G_K represented by the standard meridian-longitude system of K.



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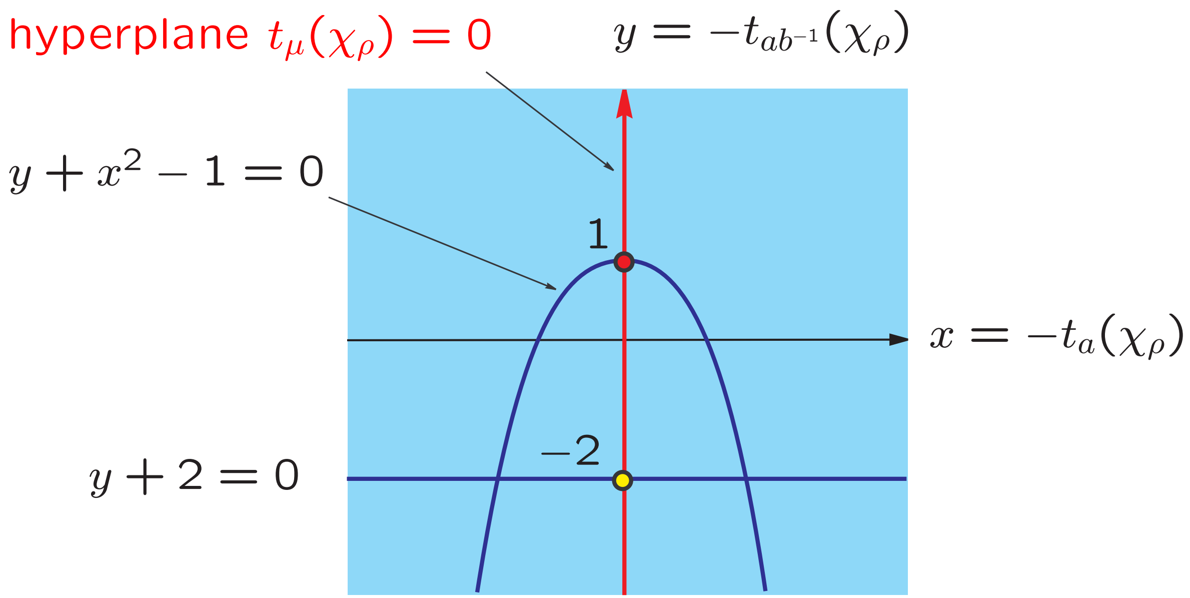
Let μ, λ be the elements in G_K represented by the standard meridian-longitude system of K.

Proposition 1.1 [N]

For any knot K, any irreducible metabelian representation $\rho : G_K \to SL(2,\mathbb{C})$ satisfies

$$\boxed{\text{trace}(\rho(\mu)) = 0}, \ \text{trace}(\rho(\lambda)) = 2.$$

EX: $X(G_{3_1})$ and its intersection with $t_\mu(\chi_\rho) = 0$



$$\boxed{S_0(K) := X(G_K) \cap \{t_\mu(\chi_\rho) = \mathrm{trace}(\rho(\mu)) = 0\}}$$

Proposition 1 says that

$$\boxed{S_0(K) \supset \{\text{irreducible metabelian characters}\}}$$

In fact, $\boxed{S_0(K) = \{\text{abel}\} \sqcup \{\text{irr. meta}\} \sqcup \{\text{irr. non-meta}\}}$

Proposition 1.2 [N] **($SL(2,\mathbb{C})$ version of [Lin], [Klassen])**

For any knot $K \subset S^3$, the number of irreducible metabelian characters of G_K is finite and given explicitly by $\frac{|\Delta_K(-1)|-1}{2}$.

EX: $S_0(3_1) = \{ \circ \, , \, \bullet \}$, $\Delta_{3_1}(t) = t - 1 + t^{-1}$, $|\Delta_{3_1}(-1)| = 3$

Corollary 1.1 [N]

If $\dim_{\mathbb{C}}(S_0(K)) = 0$, then

$$||S_0(K)|| := (\#_{\mathrm{mul}} \text{ of points in } S_0(K)) - 1 \geq \frac{|\Delta_K(-1)|-1}{2}$$

Theorem 1.1 [N]

For any 2-bridge knot $S(p,q)$,

$$||S_0(S(p,q))|| = \frac{|\Delta_{S(p,q)}(-1)|-1}{2} = \frac{p-1}{2}.$$

§1.2. Application of $S_0(K)$ to the A-polynomial $A_K(m, l)$

Reference (definition of the A-polynomial)

D. Cooper, M. Culler, H. Gillet, D. Long and P. Shalen:
Plane curves associated to character varieties of 3-manifolds,
Invent. Math. **118** (1994), 47–84.

For all $\chi_\rho \in X(G_K)$, find the points $(m, l) \in \mathbb{C}^* \times \mathbb{C}^*$ satisfying

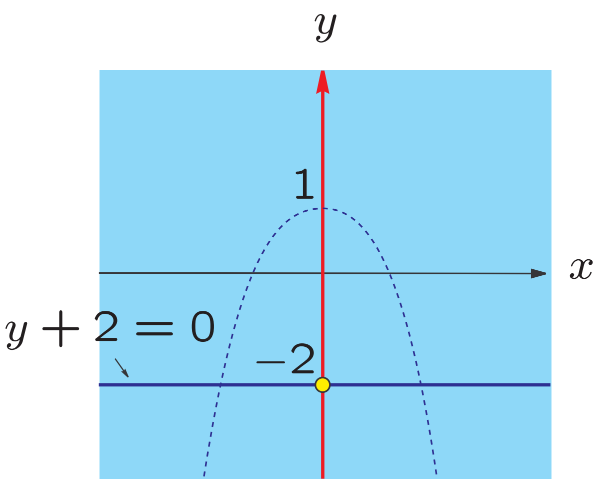
$$\boxed{(t_\mu(\chi_\rho), t_\lambda(\chi_\rho), t_{\mu\lambda}(\chi_\rho)) = (m + m^{-1}, l + l^{-1}, lm + l^{-1}m^{-1})}$$

(ignore the multiplicity of the solutions).

$$\mathcal{E}_K := \overline{\{\text{the above solutions } (m, l) \in \mathbb{C}^* \times \mathbb{C}^*\}}^{\mathbb{C}^2}$$

Then the A-polynomial $A_K(m, l)$ is defined as the defining polynomial of the 1-dimensional components of $\mathcal{E}_K - \{l = 1\}$.

EX: the A-polynomial $A_{3_1}(m,l) = lm^6 + 1$

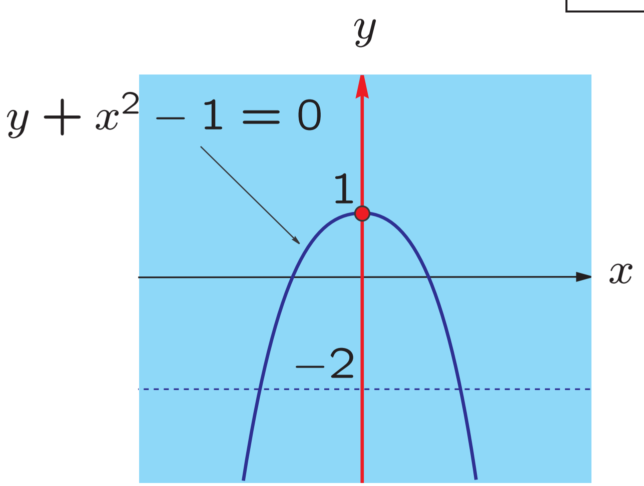


$$\begin{cases} -t_\mu(\chi_\rho) = x \\ -t_\lambda(\chi_\rho) = (-x^4+3x^2)y - 2x^4 + 6x^2 - 2 \\ -t_{\mu\lambda}(\chi_\rho) = (x^3-2x)y + 2x^3 - 3x \end{cases}$$

$$\Rightarrow (-t_\mu(\chi_\rho), -t_\lambda(\chi_\rho), -t_{\mu\lambda}(\chi_\rho)) = (x, -2, x)$$
$$= (-m - m^{-1}, -l - l^{-1}, -lm - l^{-1}m^{-1})$$

$l = 1$

discard these | ignore multiplicity | (discarding operations)



$$\Rightarrow (-t_\mu(\chi_\rho), -t_\lambda(\chi_\rho), -t_{\mu\lambda}(\chi_\rho))$$
$$= (x, x^6 - 6x^4 + 9x^2 - 2, -x^7 + 7x^5 - 14x^3 + 7x)$$
$$= (-m - m^{-1}, -l - l^{-1}, -lm - l^{-1}m^{-1})$$

$(m,l) = (\pm 1, -1)$ or $lm^6 + 1 = 0$

$$A_{3_1}(m,l) = lm^6 + 1$$

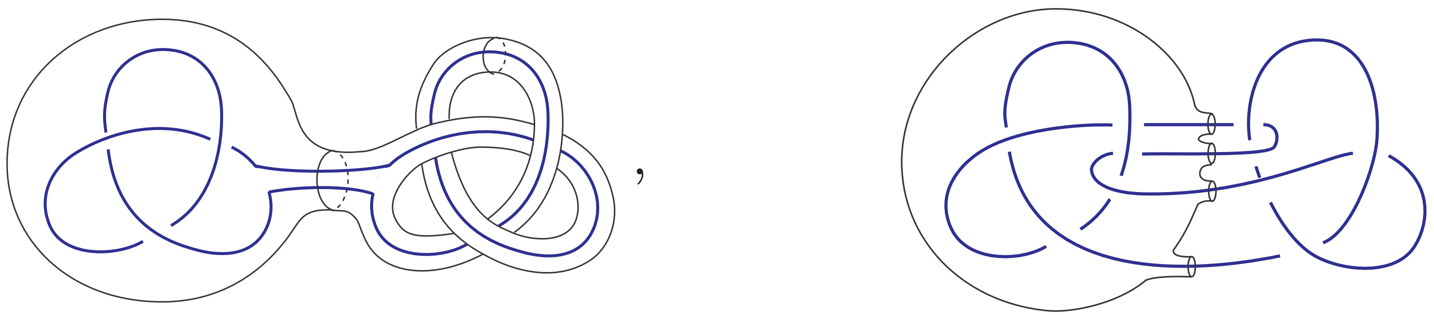
Theorem 1.2 [Le], [N]

For any 2-bridge knot $S(p,q)$,

$$\deg_l(A_{S(p,q)}(m,l)) \leq ||S_0(S(p,q))|| = \frac{p-1}{2} \text{ (best possible)}$$

(Key to Thm 2)

- $S(p,q)$ is *small* and has *no meridional boundary slope* ■



closed essential surface (swallow-follow torus), essential surface (Conway sphere)

Theorem 1.3 [N] (generalization of [Le])

For any small knot K with no meridional boundary slope,

$$\deg_l(A_K(m,l)) \leq ||S_0(K)||$$

Note. At least, 2-bridge knots and Motesinos knots with length 3 satisfies the above conditions.

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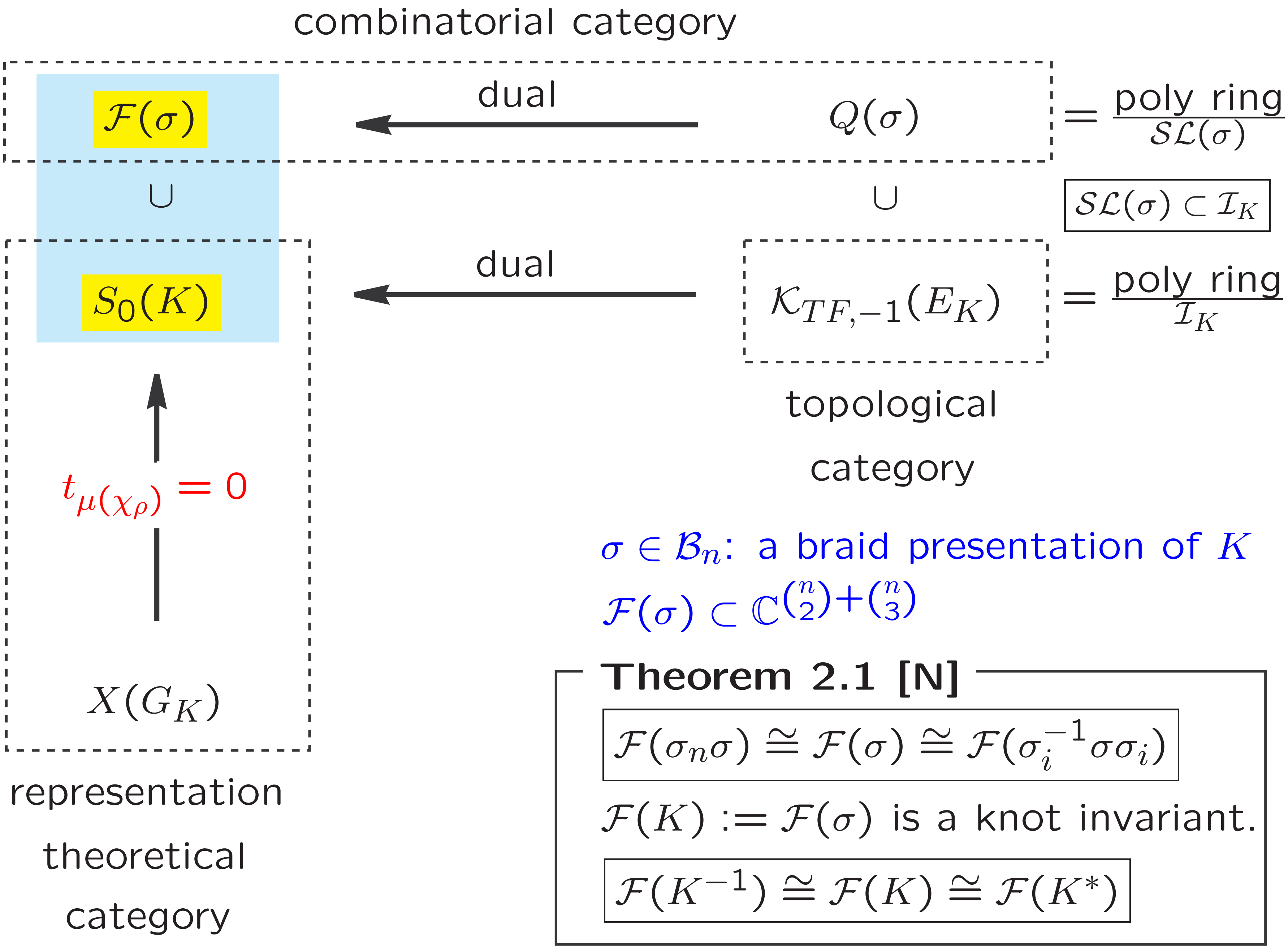


$\sigma \in \mathcal{B}_n$: a braid presentation of K

$\mathcal{F}(\sigma) \subset \mathbb{C}^{\binom{n}{2}+\binom{n}{3}}$

Theorem 2.1 [N]

$\mathcal{F}(\sigma_n\sigma) \cong \mathcal{F}(\sigma) \cong \mathcal{F}(\sigma_i^{-1}\sigma\sigma_i)$

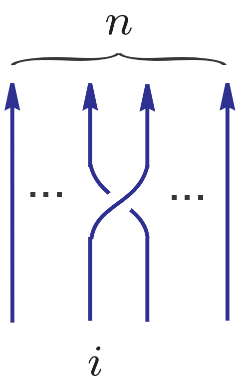
$\mathcal{F}(K) := \mathcal{F}(\sigma)$ is a knot invariant.

$\mathcal{F}(K^{-1}) \cong \mathcal{F}(K) \cong \mathcal{F}(K^*)$

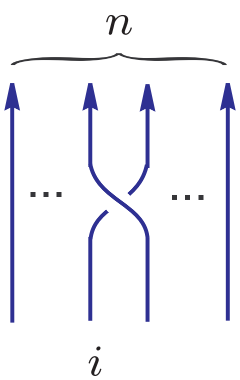
$$\mathcal{B}_n := \left\langle \sigma_1, \sigma_2, \cdots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1}, & \text{for } j = 1, 2, \cdots, n-1, \\ \sigma_j \sigma_i = \sigma_i \sigma_j, & \text{for } |i-j| \geq 2. \end{array} \right\rangle$$

Note

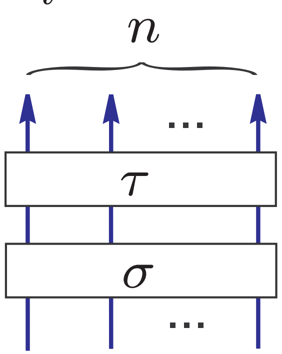
(1) $\sigma_i \leftrightarrow$



, $\sigma_i^{-1} \leftrightarrow$



(2) $\tau \cdot \sigma \leftrightarrow$



§2.1 Fundamental properties of $\mathcal{F}(K)$

Notations

$||V|| := (\underline{\#_{\mathrm{mul}} \text{ of irr. components of the variety } V}) - 1.$

$\Delta_K(t)$: the Alexander polynomial of K

$A_K(m,l)$: the A-polynomial of K.

Theorem 2.2 [N]

For a knot K with $\dim_{\mathbb{C}}(\mathcal{F}(K)) = 0$,

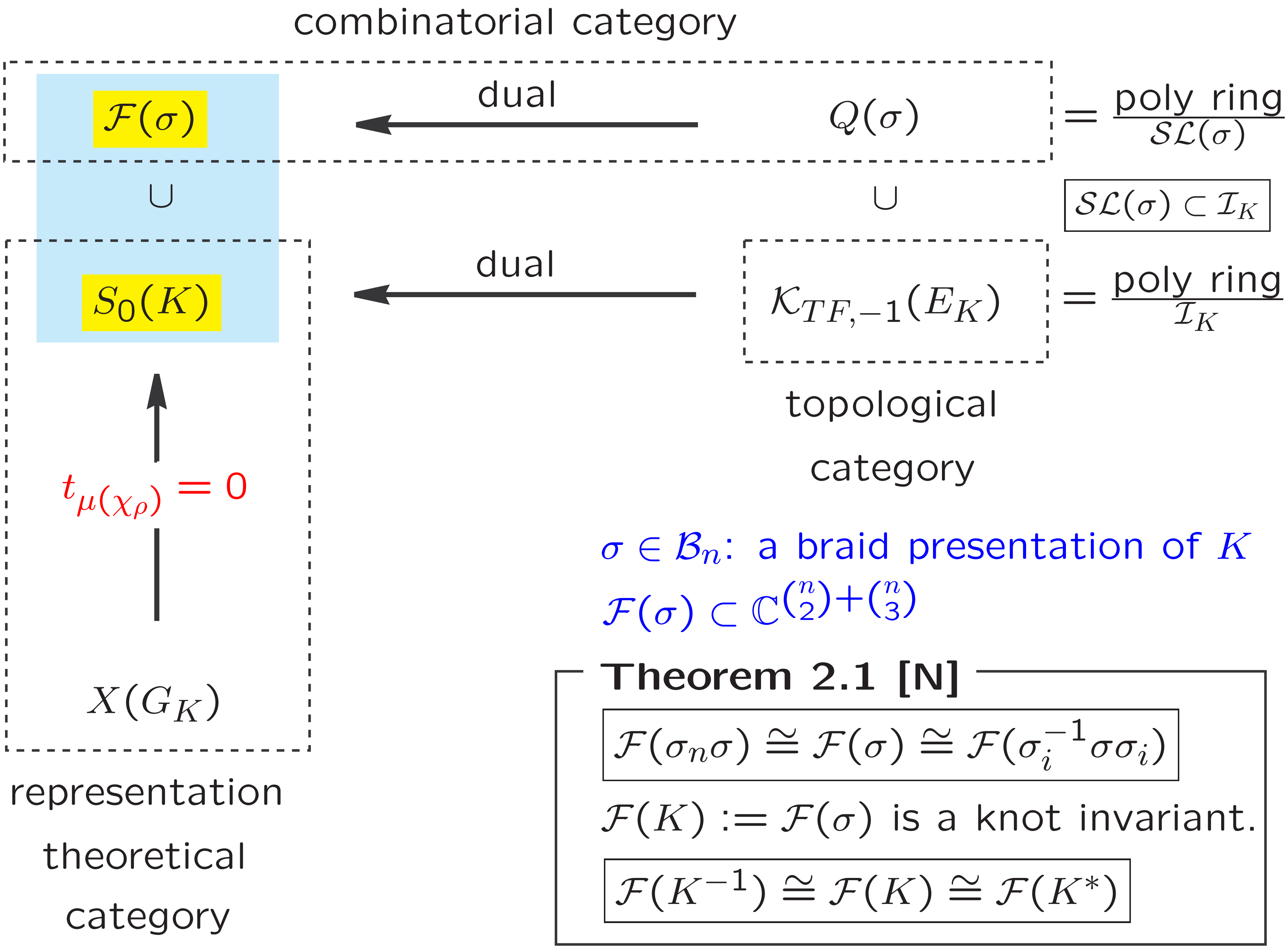
$$||\mathcal{F}(K)|| \geq ||S_0(K)|| \geq \frac{|\Delta_K(-1)|-1}{2}.$$

Theorem 2.3 [N]

If K with $\dim_{\mathbb{C}}(\mathcal{F}(K)) = 0$ is small and has no meridional ∂-slope, then $||\mathcal{F}(K)|| \geq ||S_0(K)|| \geq \deg_l(A_K(m,l))$.

K	$\dim \mathcal{F}(K)$	$\|\|\mathcal{F}(K)\|\|$	$\|\Delta_K(-1)\|$	$d_l(A_K)$	Type
O	0	0	1	0	–
3_1	0	1	3	1	S(3,1)
4_1	0	2	5	2	S(5,3)
5_1	0	2	5	1	S(5,1)
5_2	0	3	7	3	S(7,3)
6_1	0	4	9	4	S(9,5)
6_2	0	5	11	5	S(11,3)
6_3	0	6	13	6	S(13,5)
7_1	0	3	7	1	S(7,1)
7_2	0	5	11	5	S(11,5)
7_3	0	6	13	6	S(13,4)
7_4	0	7	15	5	S(15,4)
7_5	0	8	17	8	S(17,5)
7_6	0	9	19	9	S(19,7)
7_7	0	10	21	7	S(21,8)
8_1	0	6	13	6	S(13,7)
8_2	0	8	17	8	S(17,3)
8_3	0	8	17	8	S(17,13)
8_4	0	9	19	9	S(19,5)
8_5	0	12	21	9	Mont-3
8_6	0	11	23	11	S(23,7)
8_7	0	11	23	11	S(23,5)
8_8	0	12	25	12	S(25,9)

K	$\dim \mathcal{F}(K)$	$\|\|\mathcal{F}(K)\|\|$	$\|\Delta_K(-1)\|$	$d_l(A_K)$	Type
8_9	0	12	25	12	S(25,7)
8_{10}	0	15	27	???	Mont-3
8_{11}	0	13	27	11	S(27,10)
8_{12}	0	14	29	14	S(29,17)
8_{13}	0	14	29	14	S(29,12)
8_{14}	0	15	31	15	S(31,13)
8_{15}	0	18	33	???	Mont-3
8_{16}	0	17	35	???	–
8_{17}	0	18	37	???	–
8_{18}	???	???	45	???	–
8_{19}	0	3	3	???	Mont-3
8_{20}	0	6	9	5	Mont-3
8_{21}	0	9	15	???	Mont-3
9_{16}	0	21	39	???	Mont-3
9_{46}	0	6	9	4	Mont-3
10_{124}	0	4	1	???	Mont-3
10_{125}	0	9	11	???	Mont-3
10_{132}	0	8	5	8	Mont-3
10_{139}	0	7	3	4	Mont-3
10_{152}	0	13	11	???	–
10_{161}	0	10	5	???	–
$3_1 \# 3_1$	1	4	9	???	#



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$\mathcal{F}(K) := \mathcal{F}(\sigma)$ is a knot invariant.

$$\mathcal{F}(K^{-1}) \cong \mathcal{F}(K) \cong \mathcal{F}(K^*)$$

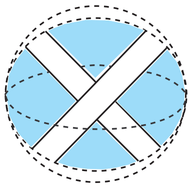
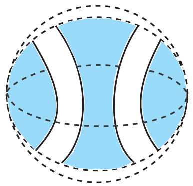
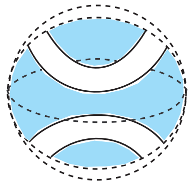
§2.2 Main tool: Kauffman bracket skein module (KBSM)

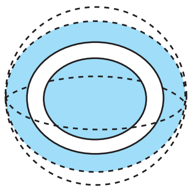
Put $\mathbb{C}_t := \mathbb{C}[t, t^{-1}]$. For a compact orientable 3-manifold M,

$$\mathcal{L}_t(M) := \mathrm{Span}_{\mathbb{C}_t}\{\text{all the isotopy classes of framed links } (\ni \phi) \text{ in } M\}$$

The KBSM $\mathcal{K}_t(M)$ of a 3-manifold M is defined as the quotient

$\mathcal{K}_t(M) := \mathcal{L}_t(M)/\langle$**Kauffman bracket skein relations at** $t\rangle$

 $- t$  $- t^{-1}$ 

$L \sqcup$  $- (-t^2 - t^{-2})L$, for any framed link L in M

Exercise: Check that in $\mathcal{K}_{-1}(M)$ the sign of a crossing and the framing of a link can be ignored. $\Rightarrow (\mathcal{K}_{-1}(M), \sqcup)$: $\mathbb{C}$-algebra

- $\mathcal{K}_{TF,-1}(E_K) = \dfrac{\mathbb{C}[x_{ij}, x_{ijk}] \ (1 \leq i < j < k \leq \exists n)}{\langle \textbf{TF-hexagon and “TF-sliding relations”} \text{ etc.} \rangle}$

TF-hexagon relations

$$x_{i_1 i_2 i_3} x_{j_1 j_2 j_3} - \frac{1}{2} \det \begin{bmatrix} x_{i_1 j_1} & x_{i_1 j_2} & x_{i_1 j_3} \\ x_{i_2 j_1} & x_{i_2 j_2} & x_{i_2 j_3} \\ x_{i_3 j_1} & x_{i_3 j_2} & x_{i_3 j_3} \end{bmatrix} \quad \begin{matrix} (1 \leq i_1 < i_2 < i_3 \leq n) \\ (1 \leq j_1 < j_2 < j_3 \leq n) \end{matrix}$$

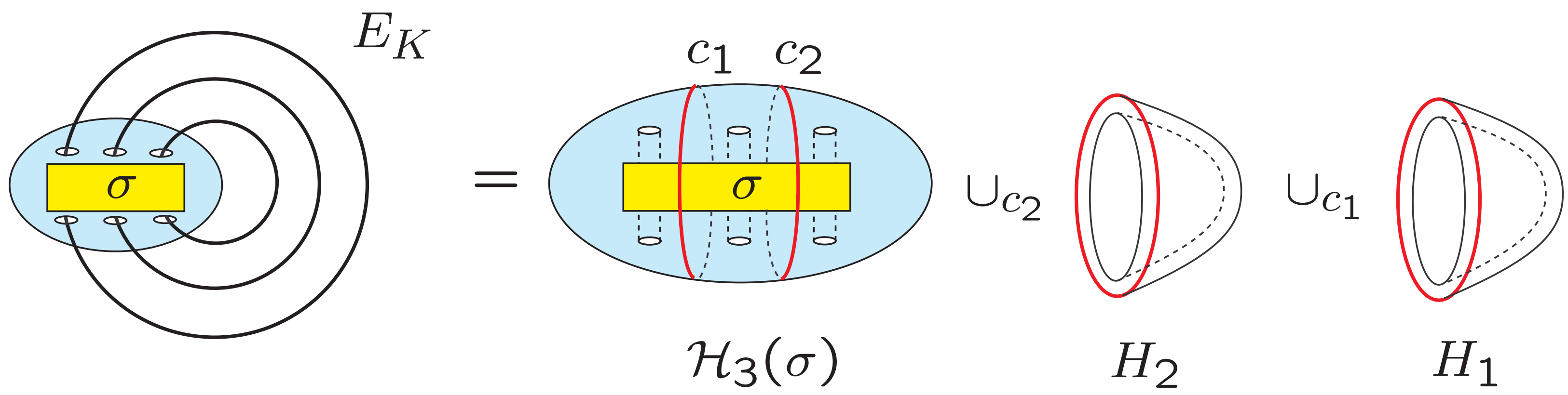
$$x_{ji} := x_{ij}, \ \ x_{ii} := 2 \ (1 \leq i \leq n)$$

- $\mathcal{K}_{TF,-1}(E_K) = \boxed{\dfrac{\mathbb{C}[x_{ij}, x_{ijk}] \ (1 \le i < j < k \le \exists n)}{\langle \textbf{TF-hexagon and ``TF-sliding relations''} \text{ etc.} \rangle}}$

1. $\boxed{\mathcal{K}_{TF,-1}(E_K) := \mathcal{K}_{-1}(E_K)/(\text{all meridional loop are zero})}$
2. $\boxed{\mathcal{K}_{-1}(E_K) = \mathcal{K}_{-1}(\mathcal{H}_n = \mathcal{H}_n(\sigma)) / \langle \textbf{sliding relations} \rangle}$

where • σ: a braid presentation of a knot K

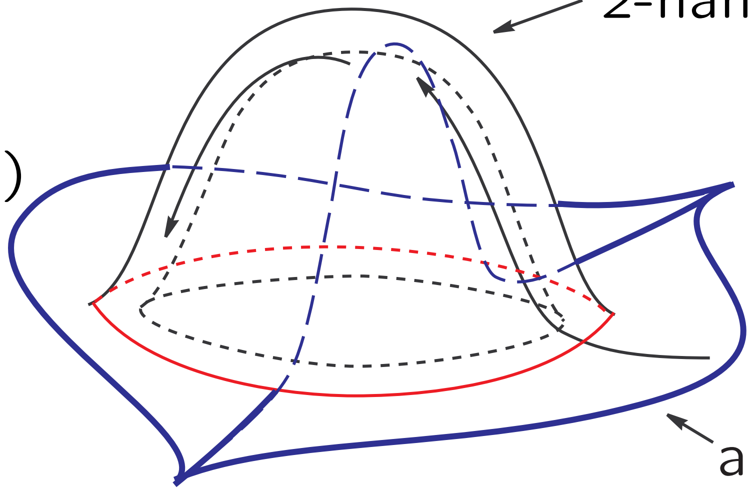
• n: the number of strings of the braid σ.



- $\mathcal{K}_{TF,-1}(E_K) = \boxed{\dfrac{\mathbb{C}[x_{ij}, x_{ijk}]\ (1 \le i < j < k \le \exists n)}{\langle \textbf{TF-hexagon and "TF-sliding relations"} \text{ etc.}\rangle}}$

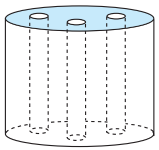
1. $\boxed{\mathcal{K}_{TF,-1}(E_K) := \mathcal{K}_{-1}(E_K)/(\text{all meridional loop are zero})}$
2. $\boxed{\mathcal{K}_{-1}(E_K) = \mathcal{K}_{-1}\left(\mathcal{H}_n = \mathcal{H}_n(\sigma)\right)/\langle \textbf{sliding relations}\rangle}$

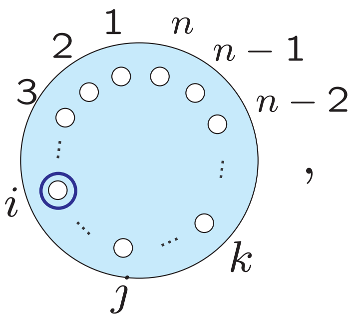
A sliding relation: $\boxed{sl(K) - K}$

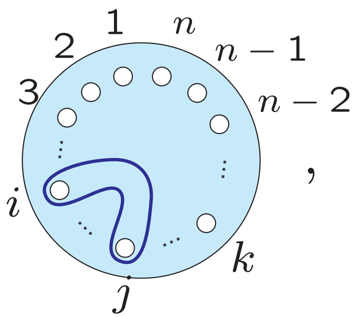


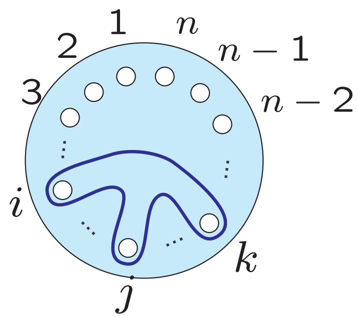
We must collect all $sl(K)$'s for all $K \subset \mathcal{H}_n$ for **sliding relations**.

- $\mathcal{K}_{TF,-1}(E_K) = \dfrac{\mathbb{C}[x_{ij}, x_{ijk}]\ (1 \leq i < j < k \leq \exists n)}{\langle \text{TF-hexagon and “TF-sliding relations” etc.} \rangle}$

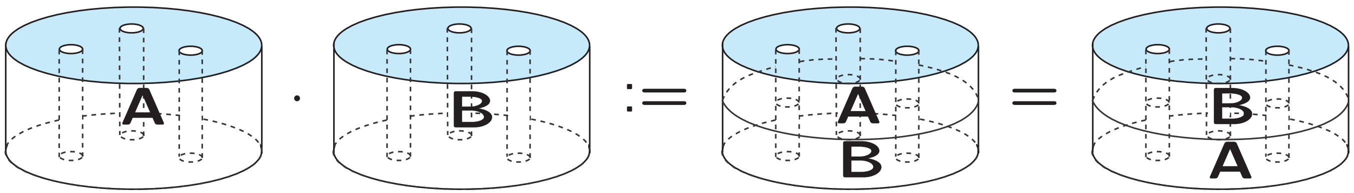
1. $\mathcal{K}_{TF,-1}(E_K) := \mathcal{K}_{-1}(E_K)/$(all meridional loop are zero)
2. $\mathcal{K}_{-1}(E_K) = \mathcal{K}_{-1}(\mathcal{H}_n)/\langle$sliding relations$\rangle$
3. $\mathcal{K}_{-1}\left(\mathcal{H}_{n\geq 3} = \right.$  $\left.\right) = \dfrac{\mathbb{C}[x_i, x_{ij}, x_{ijk}]\ (1 \leq i < j < k \leq n)}{\langle \text{“hexagon relations” etc.} \rangle}$

$x_i :=$ 

, $x_{ij} :=$ 

, $x_{ijk} :=$ 

Multiplication in $\mathcal{K}_{-1}(\mathcal{H}_n)$



$$\bullet\ \mathcal{K}_{TF,-1}(E_K) = \boxed{\frac{\mathbb{C}[x_{ij}, x_{ijk}]\ (1 \le i < j < k \le \exists n)}{\langle \textbf{TF-hexagon and “TF-sliding relations”}\ \text{etc.}\rangle}}$$

1. $\mathcal{K}_{TF,-1}(E_K) := \mathcal{K}_{-1}(E_K)/$(all meridional loop are zero)
2. $\mathcal{K}_{-1}(E_K) = \mathcal{K}_{-1}(\mathcal{H}_n)/\langle$sliding relations$\rangle$
3. $\mathcal{K}_{-1}(\mathcal{H}_{n\ge 3}) = \dfrac{\mathbb{C}[x_i, x_{ij}, x_{ijk}]\ (1 \le i < j < k \le n)}{\langle \text{“hexagon relations”}\ \text{etc.}\rangle}$

$$\begin{aligned}\mathcal{K}_{TF,-1}(E_K) &= \frac{\mathbb{C}[x_i, x_{ij}, x_{ijk}]\ (1 \le i < j < k \le n)}{\langle \textbf{“hexagon” and sliding relations},\ \boxed{x_i}\ \text{etc.}\rangle} \\ &= \frac{\mathbb{C}[x_{ij}, x_{ijk}]\ (1 \le i < j < k \le n)}{\langle \textbf{TF-hexagon and TF-sliding relations}\ \text{etc.}\rangle}\end{aligned}$$

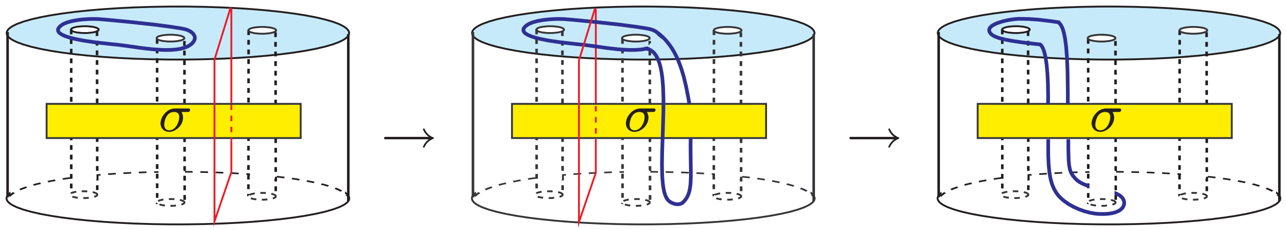
$\bullet\ \mathcal{I}_K = \langle$**TF-hexagon and TF-sliding relations** etc.$\rangle$

$\mathcal{SL}(\sigma) := \langle$TF-hexagon relations, *algebraic TF-sliding relations*$\rangle$

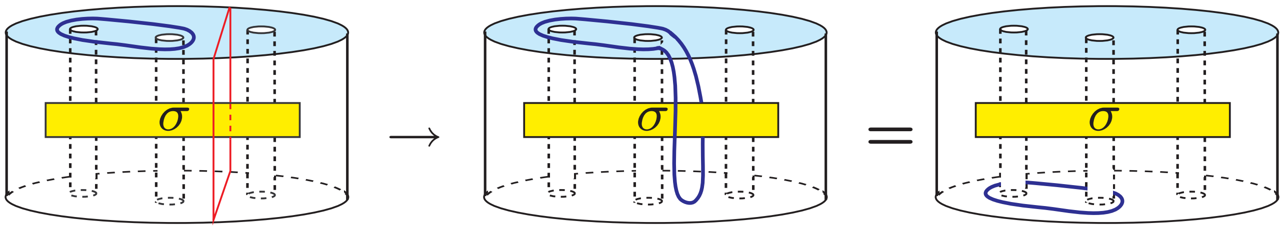
In the next subsection, we define *algebraic TF-sliding relations*.

§2.3 Construction of algebraic TF-slidings

- Type $g_{\sigma,*}$:

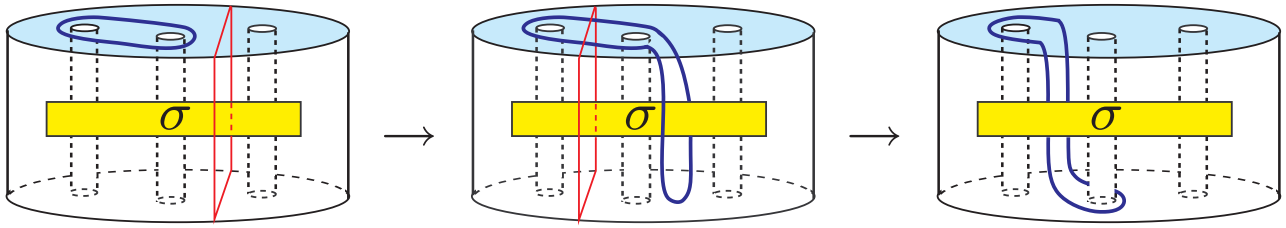


- Type f_σ:

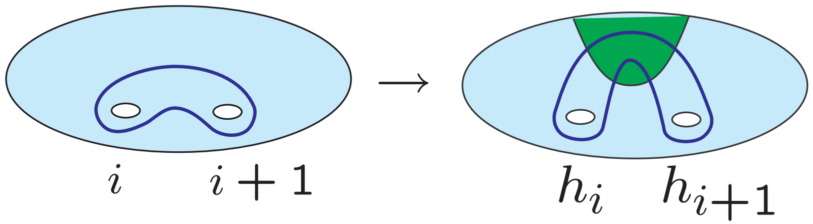


§2.3 Construction of algebraic TF-slidings

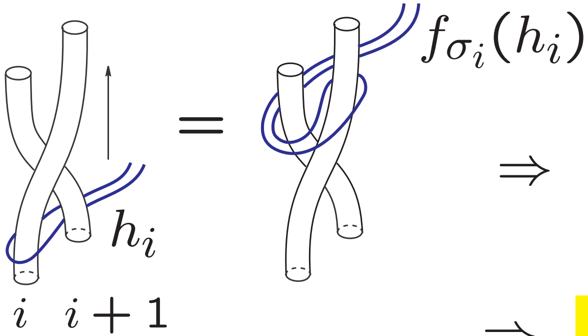
- Type $g_{\sigma,*}$:

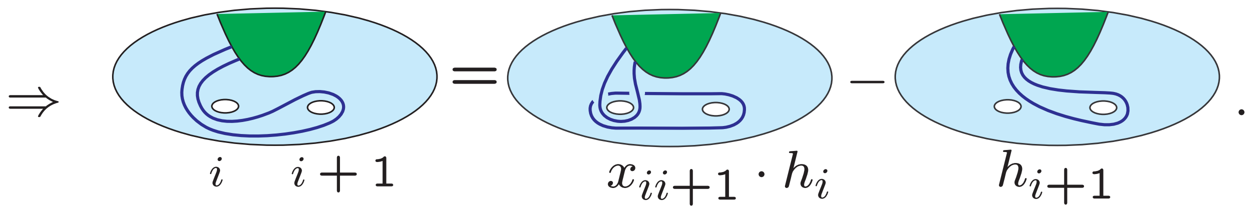


(1) Decompose $x_{i_1\cdots i_m}$ into arcs:



(2) Describe the behavior of action σ_i for h_j's:

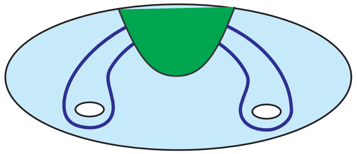




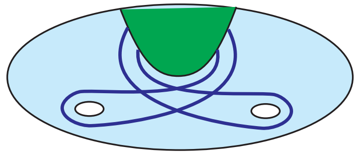
$\Rightarrow$ $f_{\sigma_i}(h_i) = x_{ii+1} \cdot h_i - h_{i+1}$.

Similarly, we get $f_{\sigma_i}(h_j) = h_{p_{i,i+1}(j)}$ for $j \neq i$, where $p_{i,i+1}$ is the permutation between i and $i+1$.

(3) Define tensor product $\otimes$ for h_i's and extend it over $\mathbb{C}[x_{ij}, x_{ijk}]$:

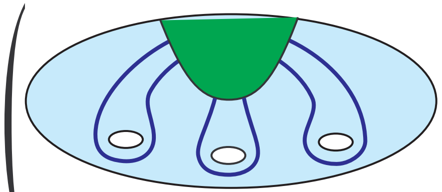
$h_i \otimes h_j :=$ 

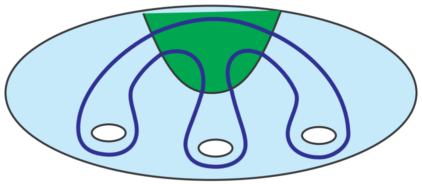
$h_i \otimes h_j$,

(**Note**: $h_j \otimes h_i =$ 

$h_j \otimes h_i$)

(4) Define a $\mathbb{C}[x_{ij}, x_{ijk}]$-linear operator c:

c (

$h_i \otimes h_j \otimes h_k$) $:=$  , $\quad c(x_{i_1 \cdots i_m}) := x_{i_1 \cdots i_m}$.

(5) Extend f_σ by $f_{\tau_1 \cdot \tau_2} := f_{\tau_1} \circ f_{\tau_2}$ ($\tau_i \in \mathcal{B}_n$) and describe $g_{\sigma,*}$, f_σ:

For a proper $A_m \subset \{1, \cdots, m\}$, $\epsilon_{A_m}(i) := \begin{cases} 1, & \text{if } i \notin A_m \\ 0, & \text{if } i \in A_m \end{cases}$

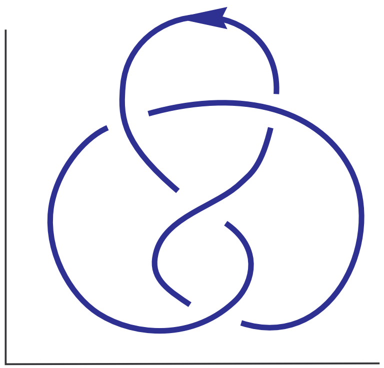
$$g_{\sigma,A_m}(h_{i_1} \otimes \cdots \otimes h_{i_m}) := f_\sigma^{\epsilon_{A_m}(1)}(h_{i_1}) \otimes \cdots \otimes f_\sigma^{\epsilon_{A_m}(m)}(h_{i_m}).$$

$$f_\sigma(h_{i_1} \otimes \cdots \otimes h_{i_m}) := f_\sigma(h_{i_1}) \otimes \cdots \otimes f_\sigma(h_{i_m})$$

$f_\sigma(\sum p_i h_i) := \sum f_\sigma(p_i) f_\sigma(h_i)$, $\quad f_\sigma(x_{i_1 \cdots i_m}) := c \circ f_\sigma(h_{i_1} \otimes \cdots \otimes h_{i_m})$

EX: $\sigma = \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$, (the closure of σ is 4_1).

figure-8 knot (4_1)



Let $a := x_{12}, b := x_{13}, c := x_{23}$.

$$
\begin{aligned}
c \circ g_{\sigma,\{2\}}(h_1 \otimes h_2) &= a^3b - a^2c - 2ab + c, \\
c \circ g_{\sigma,\{1\}}(h_1 \otimes h_2) &= b^2 - 2, \\
c \circ g_{\sigma,\{2\}}(h_1 \otimes h_3) &= a^2b^2 - 2abc + c^2 - 2, \\
c \circ g_{\sigma,\{1\}}(h_1 \otimes h_3) &= ab^4 - b^3c - 3ab^2 + 2bc + a, \\
c \circ g_{\sigma,\{2\}}(h_2 \otimes h_3) &= b, \\
c \circ g_{\sigma,\{1\}}(h_2 \otimes h_3) &= ab^3c - a^2b^2 - b^2c^2 + 2, \\
c \circ g_{\sigma,\{1\}}(h_1 \otimes h_1) &= -a^2b + ac + b, \\
c \circ g_{\sigma,\{2\}}(h_2 \otimes h_2) &= -bc + a, \\
c \circ g_{\sigma,\{3\}}(h_3 \otimes h_3) &= -ab^3 + b^2c + 2ab - c, \\
c \circ f_{\sigma}(h_1 \otimes h_2) &= a^2b^3 - 2ab^2c - a^2b + bc^2 + ac - b, \\
c \circ f_{\sigma}(h_1 \otimes h_3) &= a^3b^5 - 3a^2b^4c - 2a^3b^3 + 3ab^3c^2 + 4a^2b^2c \\
&\quad -b^2c^3 + a^3b - ab^3 - 2abc^2 - a^2c + b^2c + c, \\
c \circ f_{\sigma}(h_2 \otimes h_3) &= ab^2 - bc - a.
\end{aligned}
$$

§2.4 Construction of $\mathcal{SL}(\sigma)$ and $\mathcal{F}(\sigma)$ for $\sigma \in \mathcal{B}_n$

- $\sigma \in \mathcal{B}_n$: a braid presentation of a knot $K \subset S^3$.

$\mathcal{SL}(\sigma) \subset \mathbb{C}[x_{ij}, x_{ijk}]$ is defined as the ideal generated by

$$c\Big(f_\sigma(h_{i_1} \otimes h_{i_2}) - h_{i_1} \otimes h_{i_2}\Big), \quad \text{for } 1 \le i_1 < i_2 \le n,$$
$$c\Big(g_{\sigma,*}(h_{i_1} \otimes h_{i_2}) - h_{i_1} \otimes h_{i_2}\Big), \quad \text{for } 1 \le i_1 \le i_2 \le n,$$
$$c\Big(f_\sigma(h_{i_1} \otimes h_{i_2} \otimes h_{i_3}) - h_{i_1} \otimes h_{i_2} \otimes h_{i_3}\Big), \quad \text{for } 1 \le i_1 < i_2 < i_3 \le n,$$
$$c\Big(g_{\sigma,*}(h_{i_1} \otimes h_{i_2} \otimes h_{i_3}) - h_{i_1} \otimes h_{i_2} \otimes h_{i_3}\Big), \quad \text{for } 1 \le i_1 \le i_2 \le i_3 \le n, \text{ except } i_1 = i_2 = i_3.$$

TF-hexagon relations

(“$*$” runs through all proper subsets of $\{1,2,3\}$)

Then $Q(\sigma) := \mathbb{C}[x_{ij}, x_{ijk}]/\mathcal{SL}(\sigma)$.

Definition 2.1 (algebraic variety $\mathcal{F}(\sigma)$)

$\mathcal{F}(K) := \mathcal{F}(\sigma)$ is the common zeros of $\mathcal{SL}(\sigma)$ with multiplicity.

§2.5 Duality of $\mathcal{K}_{TF,-1}(E_K)$ and $S_0(K)$

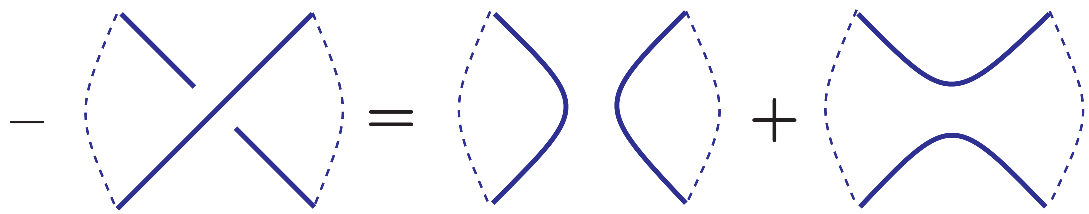
What is the ideal $\mathcal{I}_K$?

(1) $\boxed{t_{x^{-1}} = t_x}$ $\Rightarrow$ ignore the orientation of the loop x

(2) $\boxed{t_{yxy^{-1}} = t_x}$ $\Rightarrow$ ignore the base point

(3) $\boxed{\text{trace-identity}}$ $\Leftrightarrow$ skein relation at $t = -1$

$t_{xy} = t_x t_y - t_{xy^{-1}} \leftrightarrow$



$-t_{[K]} \leftrightarrow K$

$\boxed{\{\text{equations in } t_x \text{ derived from relations of } G_K \text{ via trace-id \& } x_i = 0\} = \mathcal{I}_K}$

$S_0(K) = \mathrm{Sol}_{\mathbb{C}}^{\mathrm{mul}}(\mathcal{I}_K) \subset \mathrm{Sol}_{\mathbb{C}}^{\mathrm{mul}}(\mathcal{SL}(\sigma)) = \mathcal{F}(K)$ ■

§2.6 Filtration of $Q(\sigma)$ and sequence $\{\mathcal{F}^{(d)}(K)\}_{d=1,2,3}$

For a knot K, let $\sigma \in \mathcal{B}_n$ be a braid presentation of K.

In fact, $Q(\sigma)$ has a filtration:

$$\begin{array}{ccccccc}
Q(\sigma) = Q^{(3)}(\sigma) & \supset & Q^{(2)}(\sigma) & \supset & Q^{(1)}(\sigma) = \mathbb{C} \\
\text{dual} \downarrow & & \downarrow & & \downarrow \\
\mathcal{F}(\sigma) = \mathcal{F}^{(3)}(\sigma) & \to & \mathcal{F}^{(2)}(\sigma) & \to & \mathcal{F}^{(1)}(\sigma) = \{0\} \\
\cap & & \cap & & \cap \\
\mathbb{C}^{\binom{n}{2}+\binom{n}{3}} & \xrightarrow{\text{proj}} & \mathbb{C}^{\binom{n}{2}} & \xrightarrow{\text{proj}} & \mathbb{C}
\end{array}$$

- $Q^{(d)}(\sigma) := \mathcal{C}_n^{(d)} / \mathcal{SL}^{(d)}(\sigma)$

where $\boxed{\mathcal{C}_n^{(3)} := \mathbb{C}[x_{ij}, x_{ijk}],\ \mathcal{C}_n^{(2)} := \mathbb{C}[x_{ij}],\ \mathcal{C}_n^{(1)} := \mathbb{C}}$

§2.6 Filtration of $Q(\sigma)$ and sequence $\{\mathcal{F}^{(d)}(K)\}_{d=1,2,3}$

For a knot K, let $\sigma \in \mathcal{B}_n$ be a braid presentation of K.

In fact, $Q(\sigma)$ has a filtration:

$$\begin{array}{ccccccc} Q(\sigma) = Q^{(3)}(\sigma) & \supset & Q^{(2)}(\sigma) & \supset & Q^{(1)}(\sigma) = \mathbb{C} \\ \text{dual} \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(\sigma) = \mathcal{F}^{(3)}(\sigma) & \to & \mathcal{F}^{(2)}(\sigma) & \to & \mathcal{F}^{(1)}(\sigma) = \{0\} \\ \cap & & \cap & & \cap \\ \mathbb{C}^{\binom{n}{2}+\binom{n}{3}} & \xrightarrow{\text{proj}} & \mathbb{C}^{\binom{n}{2}} & \xrightarrow{\text{proj}} & \mathbb{C} \end{array}$$

- $Q^{(d)}(\sigma) := \mathcal{C}_n^{(d)} / \mathcal{SL}^{(d)}(\sigma)$
- $\mathcal{SL}^{(2)}(\sigma)$ be defined as the ideal of $\mathcal{C}_n^{(2)}$ generated by

$$c\Big(f_\sigma(h_{i_1} \otimes h_{i_2}) - h_{i_1} \otimes h_{i_2}\Big), \quad \text{for } 1 \leq i_1 < i_2 \leq n,$$
$$c\Big(g_{\sigma,*}(h_{i_1} \otimes h_{i_2}) - h_{i_1} \otimes h_{i_2}\Big), \quad \text{for } 1 \leq i_1 \leq i_2 \leq n,\ * = \{1\} \text{ or } \{2\},$$

- $\mathcal{SL}^{(1)}(\sigma)$ is defined as zero ideal $\langle 0 \rangle \subset \mathbb{C}$.

§2.6 Filtration of $Q(\sigma)$ and sequence $\{\mathcal{F}^{(d)}(K)\}_{d=1,2,3}$

For a knot K, let $\sigma \in \mathcal{B}_n$ be a braid presentation of K.

In fact, $Q(\sigma)$ has a filtration:

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Q(\sigma) = Q^{(3)}(\sigma) & \supset & Q^{(2)}(\sigma) & \supset & Q^{(1)}(\sigma) = \mathbb{C} \\
\text{dual} \downarrow & & \downarrow & & \downarrow \\
\mathcal{F}(\sigma) = \mathcal{F}^{(3)}(\sigma) & \to & \mathcal{F}^{(2)}(\sigma) & \to & \mathcal{F}^{(1)}(\sigma) = \{0\} \\
\cap & & \cap & & \cap \\
\mathbb{C}^{\binom{n}{2}+\binom{n}{3}} & \xrightarrow{\text{proj}} & \mathbb{C}^{\binom{n}{2}} & \xrightarrow{\text{proj}} & \mathbb{C}
\end{array}$$

Theorem 2.4 [N]

For $d = 1, 2, 3$, $\mathcal{F}^{(d)}(K) := \mathcal{F}^{(d)}(\sigma)$ are knot invariants (up to isom $\cong$).

$\mathcal{F}^{(1)}(K)$: trivial inv. In fact, $\mathcal{F}^{(d)}(K^{-1}) \cong \mathcal{F}^{(d)}(K) \cong \mathcal{F}^{(d)}(K^{*})$.

K	$\dim \mathcal{F}(K)$	$\dim \mathcal{F}^{(2)}(K)$	$\|\|\mathcal{F}(K)\|\|$	$\|\|\mathcal{F}^{(2)}(K)\|\|$	$\|\Delta_K(-1)\|$	$d_l(A_K)$	Type
O	0	0	0	0	1	0	–
3_1	0	0	1	1	3	1	S(3,1)
4_1	0	0	2	2	5	2	S(5,3)
5_1	0	0	2	2	5	1	S(5,1)
5_2	0	0	3	3	7	3	S(7,3)
6_1	0	0	4	4	9	4	S(9,5)
6_2	0	0	5	5	11	5	S(11,3)
6_3	0	0	6	6	13	6	S(13,5)
7_1	0	0	3	3	7	1	S(7,1)
7_2	0	0	5	5	11	5	S(11,5)
7_3	0	0	6	6	13	6	S(13,4)
7_4	0	0	7	7	15	5	S(15,4)
7_5	0	0	8	8	17	8	S(17,5)
7_6	0	0	9	9	19	9	S(19,7)
7_7	0	0	10	10	21	7	S(21,8)
8_1	0	0	6	6	13	6	S(13,7)
8_2	0	0	8	8	17	8	S(17,3)
8_3	0	0	8	8	17	8	S(17,13)
8_4	0	0	9	9	19	9	S(19,5)
8_5	0	0	12	11	21	9	Mont-3
8_6	0	0	11	11	23	11	S(23,7)
8_7	0	0	11	11	23	11	S(23,5)
8_8	0	0	12	12	25	12	S(25,9)

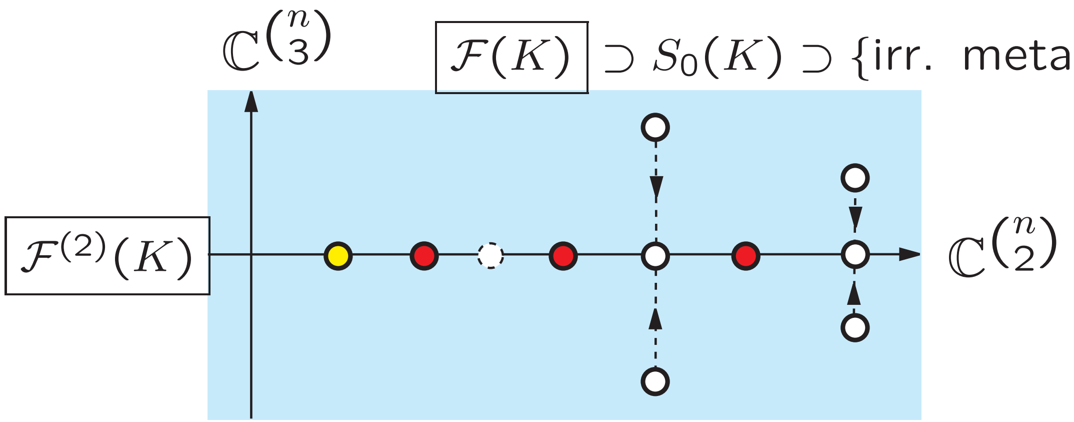
K	$\dim \mathcal{F}(K)$	$\dim \mathcal{F}^{(2)}(K)$	$\|\|\mathcal{F}(K)\|\|$	$\|\|\mathcal{F}^{(2)}(K)\|\|$	$\|\Delta_K(-1)\|$	$d_l(A_K)$	Type
8_9	0	0	12	12	25	12	S(25,7)
8_{10}	0	0	15	14	27	???	Mont-3
8_{11}	0	0	13	13	27	11	S(27,10)
8_{12}	0	0	14	14	29	14	S(29,17)
8_{13}	0	0	14	14	29	14	S(29,12)
8_{14}	0	0	15	15	31	15	S(31,13)
8_{15}	0	0	18	17	33	???	Mont-3
8_{16}	0	0	17	17	35	???	–
8_{17}	0	0	18	18	37	???	–
8_{18}	???	???	???	???	45	???	–
8_{19}	0	0	3	2	3	???	Mont-3
8_{20}	0	0	6	5	9	5	Mont-3
8_{21}	0	0	9	8	15	???	Mont-3
9_{16}	0	0	21	20	39	???	Mont-3
9_{46}	0	0	6	5	9	4	Mont-3
10_{124}	0	0	4	2	1	???	Mont-3
10_{125}	0	0	9	7	11	???	Mont-3
10_{132}	0	0	8	5	5	8	Mont-3
10_{139}	0	0	7	4	3	4	Mont-3
10_{152}	0	0	13	9	11	???	–
10_{161}	0	0	10	6	5	???	–
$3_1\#3_1$	1	1	4	3	9	???	#

Question. $\mathcal{F}(K) = S_0(K)$?

Theorem 2.5 [N]

If $\dim(\mathcal{F}^{(2)}(K)) = 0$ and $||\mathcal{F}^{(2)}(K)|| = \frac{|\Delta_K(-1)|-1}{2}$, then

$$\mathcal{F}^{(2)}(K) = \mathcal{F}(K) = S_0(K).$$



Corollary 2.1 [N]

For any 2-bridge knot $S(p,q)$, $\mathcal{F}(S(p,q)) = S_0(S(p,q))$.

In particular, $||\mathcal{F}(S(p,q))|| = ||S_0(S(p,q))|| = \frac{p-1}{2}$.

Checked: $\mathcal{F}(8_{16}) = S_0(8_{16})$, $\mathcal{F}(8_{17}) = S_0(8_{17})$, $\mathcal{F}(10_{132}) = S_0(10_{132})$.

Contents

§3.1 Definition of abelian knot contact homology $HC_*^{\text{ab}}(K)$

(Due to L. Ng.) For a knot K, fix a braid presentation $\sigma \in \mathcal{B}_n$.

$C_n := \mathbb{Z}[\overbrace{a_{ij}}^{\deg 0}, \overbrace{b_{ij}, c_{ij}}^{\deg 1}, \overbrace{d_{ij}, e_i}^{\deg 2}]$ $(1 \leq i < j \leq n)$, $\boxed{vw = (-1)^{\deg v \deg w} wv}$.

$$C_n = C_n^{(2)} \supset C_n^{(1)} \supset C_n^{(0)} = \mathbb{Z}[a_{ij}]$$

$$0 \to C_n^{(2)} \xrightarrow{\partial_\sigma^{(2)}} C_n^{(1)} \xrightarrow{\partial_\sigma^{(1)}} C_n^{(0)} \xrightarrow{\partial_\sigma^{(0)}} 0,$$

where $\partial_\sigma^{(i)}$ is a certain differential satisfying the followings:

- $\partial_\sigma^{(i)}$ lowers degree by one
- the Leibniz rule $\partial_\sigma^{(i)}(vw) = \left(\partial_\sigma^{(i)} v\right) w + (-1)^{\deg v} v \left(\partial_\sigma^{(i)} w\right)$
- $\partial_\sigma^{(i)} \circ \partial_\sigma^{(i+1)} = 0$

$$\boxed{HC_i^{\text{ab}}(\sigma) := \text{Ker}\partial_\sigma^{(i)} / \text{Im}\partial_\sigma^{(i+1)}}$$

§3.2 $HC_0^{\mathrm{ab}}(K)$ and $\mathcal{F}^{(2)}(K)$

For a knot K, fix a braid presentation $\sigma \in \mathcal{B}_n$.

$$0 \to C_n^{(2)} \xrightarrow{\partial_\sigma^{(2)}} C_n^{(1)} \xrightarrow{\partial_\sigma^{(1)}} C_n^{(0)} \xrightarrow{\partial_\sigma^{(0)}} 0,$$

$$\boxed{HC_i^{\mathrm{ab}}(\sigma) := \mathrm{Ker}\partial_\sigma^{(i)}/\mathrm{Im}\partial_\sigma^{(i+1)}}$$

Theorem 3.1 [Ng]

$HC_0^{\mathrm{ab}}(\sigma)$ is invariant under the Markov move type I and II. Hence $HC_0^{\mathrm{ab}}(K) := HC_0^{\mathrm{ab}}(\sigma)$ is an invariant of knots.

Theorem 3.2 [N]

For any knot $K \subset S^3$ with an arbitrary braid presentation $\sigma \in \mathcal{B}_n$, there exist an isomorphism

$$\Phi : HC_0^{\mathrm{ab}}(\sigma; \mathbb{C}) \to Q^{(2)}(\sigma), \quad \Phi(a_{ij}) = -x_{ij}.$$

References

For more information on this topic...

- F. Nagasato: *Algebraic varieties via a filtration of the Kauffman bracket skein module and knot contact homology*, preprint. This preprint will be available at `http://www.math.titech.ac.jp/~fukky/`.

- F. Nagasato: *Finiteness of a section of the $SL(2,\mathbb{C})$-character variety of knot groups*, to appear, avalable at `http://www.math.titech.ac.jp/~fukky/`.

- L. Ng: *Knot and braid invariants from contact homology I, II*, Geom. Topol. **9** (2005), 247–297. Geom. Topol. **9** (2005), 1603-1637.