ALGEBRAIC EQUATIONS AND KNOT INVARIANTS

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1. Introduction

This is a report on the talk which the author gave at the conference *Topology of knots IX (2006)* at Nihon University, Japan.

In this report, we introduce algebraic varieties $\mathcal{F}^{(d)}(K)$ $(d = 1, 2, 3)$ in a complex space $\mathbb{C}^N$ defined for an oriented knot K in 3-sphere S^3. The varieties $\mathcal{F}^{(d)}(K)$ in fact give some viewpoints to researches for the knot theory, for example, *Fox's coloring*[1] from the representation theoretical viewpoint, a *shortcut*[2] to the $SL(2,\mathbb{C})$-character variety of knot groups introduced by Culler and Shalen [CS], and so on. The idea of the latter viewpoint is actually based on the researches of X.-S. Lin [L] and the author [N2]. So one may think of $\mathcal{F}^{(d)}$ $(d = 1, 2, 3)$ especially the third variety $\mathcal{F}^{(3)}(K)$ as a kind of *generalization* of these researches.

Now, the varieties $\mathcal{F}^{(d)}(K)$ $(d = 1, 2, 3)$ are defined in the following steps. For a braid presentation σ of a knot K, we first construct finitely many polynomials on $\mathbb{C}^N$ by using an action of the braid σ on the Kauffman bracket skein module (KBSM) of a handlebody at $t = -1$ with *trace-free condition*. Then the ideal $\mathcal{SL}^{(3)}(\sigma)$ generated by the polynomials gives an algebraic variety $\mathcal{F}^{(3)}(\sigma)$ via the Hilbert Nullstellensatz. In fact, $\mathcal{F}^{(3)}(\sigma)$ turns out to be invariant under the Markov moves and thus becomes a knot invariant. This is a desired variety $\mathcal{F}^{(3)}(K)$. The above process can be used for *restrictions* $\mathcal{SL}^{(2)}(\sigma)$ and $\mathcal{SL}^{(1)}(\sigma)$ of the ideal $\mathcal{SL}^{(3)}(\sigma)$. Therefore we can get knot invariants $\mathcal{F}^{(d)}(K)$ $(d = 1, 2, 3)$.

The first variety $\mathcal{F}^{(1)}(K)$ is actually a trivial invariant. The third one $\mathcal{F}^{(3)}(K)$ can be considered as a variety containing a *section* of the $SL(2,\mathbb{C})$-character variety of the knot group by using Bullock's theorem (quantization of the $SL(2,\mathbb{C})$-character variety). This view point gives relationships of the variety $\mathcal{F}^{(3)}(K)$ with the number of $SL(2,\mathbb{C})$-irreducible metabelian characters of the knot group (the knot determinant), and moreover the maximal degree (or *span*) of the A-polynomial $A_K(m,l)$ in terms of l, which polynomial is a knot invariant introduced by Cooper, Culler, Gillet, Long and Shalen [CCGLS]. Regarding the second variety $\mathcal{F}^{(2)}(K)$, the quotient ring $\mathbb{C}[x_1, \cdots, x_n]/\mathcal{SL}^{(2)}(\sigma)$ $(n \leq N)$ turns out to be isomorphic to the degree 0 knot contact homology which was researched by L. Ng in detail.

In this report, we show a sketch of a proof of the above statement. For more information, please refer to [N1].

The author has been supported by JSPS Research Fellowships for Young Scientists.

[1] Since the topic is not the main object in this report, we omit the reason why $\mathcal{F}^{(d)}(K)$ and the coloring are related. We will discuss the topic in another paper.

[2] i.e., a combinatorial realization of the character variety: we can calculate directly the character variety from a diagram of the knot.

2. Algebraic varieties $\mathcal{F}^{(d)}(\sigma)$ $(d = 1, 2, 3)$

Let $\sigma \in \mathcal{B}_n$ ($n \geq 3$ for convenience) is a braid presentation of a knot $K \subset S^3$. For $1 \leq d \leq 3$, let $\mathcal{C}_n^{(1)} := \mathbb{C}$, $\mathcal{C}_n^{(d)} := \mathbb{C}[\{x_{i_1 \cdots i_m}\}_{1 \leq i_1 < \cdots < i_m \leq n,\ 2 \leq m \leq d}]$ $(d = 2, 3)$. (Note that $\mathcal{C}_n^{(1)} \subset \mathcal{C}_n^{(2)} \subset \mathcal{C}_n^{(3)}$.) Then for $1 \leq d \leq 3$, let $\mathcal{A}_n^{(d)} := \bigoplus_{i=1}^{d} \left(\bigoplus_{j=1}^{n} \mathcal{C}_n^{(3)} h_j \right)^{\otimes i}$, where $\otimes$ is taken over $\mathcal{C}_n^{(3)}$. We define a homomorphism $c : \mathcal{A}_n^{(d)} \to \mathcal{C}_n^{(3)}$ as $\mathcal{C}_n^{(3)}$-module by

- $c(h_i) := 0$
- $c(h_{i_1} \otimes h_{i_2}) := \begin{cases} 2, & \text{if } i_1 = i_2 \\ x_{i_{\tau(1)} i_{\tau(2)}}, & \text{otherwise} \end{cases}$
- $c(h_{i_1} \otimes h_{i_2} \otimes h_{i_3}) := \mathrm{sign}(\tau) x_{i_{\tau(1)} i_{\tau(2)} i_{\tau(3)}}$

where $\tau \in \mathcal{S}_2$ such that $i_{\tau(1)} < i_{\tau(2)}$ and $\tau \in \mathcal{S}_3$ such that $i_{\tau(1)} \leq i_{\tau(2)} \leq i_{\tau(3)}$. Then we can define *twisted* automorphisms

$$f_\sigma : \mathcal{A}_n^{(d)} \to \mathcal{A}_n^{(d)}, \qquad g_{\sigma, A} : \mathcal{A}_n^{(d)} \to \mathcal{A}_n^{(d)},$$

for $\sigma \in \mathcal{B}_n$ and a subset $A \subset \{1, \cdots, d\}$.

Definition 2.1 (**twisted automorphism f_σ of $\mathcal{A}_n^{(d)}$).**

$$f_{\sigma_i}(h_j) := \begin{cases} x_{ii+1} \cdot h_j - h_{p_{i,i+1}(j)}, & \textit{if } j = i, \\ h_{p_{i,i+1}(j)}, & \textit{otherwise}, \end{cases}$$

$$f_{\sigma_i^{-1}}(h_j) := \begin{cases} x_{ii+1} \cdot h_j - h_{p_{i,i+1}(j)}, & \textit{if } j = i+1, \\ h_{p_{i,i+1}(j)}, & \textit{otherwise}, \end{cases}$$

where p_{ii+1} is the permutation between i and $i+1$. Then $f_{\sigma_i^{\pm}}$ are extended by

$$f_{\tau_1 \cdot \tau_2}(h_i) := f_{\tau_1} \circ f_{\tau_2}(h_i), \ \textit{for } \tau_1, \tau_2 \in \mathcal{B}_n,$$

$$f_\sigma(h_{i_1} \otimes \cdots \otimes h_{i_m}) := f_\sigma(h_{i_1}) \otimes \cdots \otimes f_\sigma(h_{i_m}),$$

$$f_\sigma(x_{i_1 \cdots i_m}) := c \circ f_\sigma(h_{i_1} \otimes \cdots \otimes h_{i_m}).$$

Here we have a remark on the twisted homomorphism restricted to the subring $\mathcal{C}_n^{(2)}$ of the coefficient ring $\mathcal{C}_n^{(3)}$. For any $\sigma \in \mathcal{B}_n$, the automorphism $f_\sigma : \mathcal{C}_n^{(2)} \to \mathcal{C}_n^{(2)}$ satisfies

$$f_{\sigma_i}(x_{jk}) := \begin{cases} x_{ii+1} \cdot x_{jk} - x_{p_{i,i+1}(jk)}, & \text{if } i \in \{j,k\},\ i+1 \notin \{j,k\}, \\ x_{p_{i,i+1}(jk)}, & \text{otherwise}, \end{cases}$$

$$f_{\sigma_i^{-1}}(x_{jk}) := \begin{cases} x_{ii+1} \cdot x_{jk} - x_{p_{i,i+1}(jk)}, & \text{if } i+1 \in \{j,k\},\ i \notin \{j,k\}, \\ x_{p_{i,i+1}(jk)}, & \text{otherwise}. \end{cases}$$

We consider the homomorphism $\Psi : \mathcal{B}_n \to \mathrm{Aut}(\mathcal{C}_n^{(2)})$ defined by $\Psi(\sigma) := f_\sigma$. Then the composition $\Psi \circ \widetilde{i}$ of the homomorphism Ψ with the twisted inclusion $\widetilde{i} : \mathcal{B}_n \to \mathcal{B}_{n+1}$, $\widetilde{i}(\sigma) := \sigma^{-1}$, is actually *the Magnus representation* of the braid group $\mathcal{B}_n$ introduced in the paper [M].

Definition 2.2 (**twisted automorphism** $g_{\sigma,A}$ **of** $\mathcal{A}_n^{(d)}$)**.** *For a non-empty subset* $A \subset \{1,2,3\}$,

$$\epsilon(i) := \begin{cases} 0 & \textit{if } i \in A \\ 1 & \textit{otherwise} \end{cases}$$

Then the twisted automorphism $g_{\sigma,A}$ *of* $\mathcal{A}_n^{(d)}$ *is defined by*

$$g_{\sigma,A}(h_{i_1} \otimes \cdots \otimes h_{i_m}) := f_\sigma^{\epsilon(1)}(h_{i_1}) \otimes \cdots \otimes f_\sigma^{\epsilon(m)}(h_{i_m})$$

$$g_{\sigma,A}(x_{i_1 \cdots i_m}) := c \circ g_{\sigma,A}(h_{i_1} \otimes \cdots \otimes h_{i_m})$$

Now, we are ready to define the ideal $\mathcal{SL}^{(d)}(\sigma)$ $(d = 1,2,3)$. First we define $\mathcal{SL}^{(d)}(\sigma)$ for $d = 3$. Let $\mathcal{SL}^{(3)}(\sigma)$ be the ideal of $\mathcal{C}_n^{(3)}$ generated by

$$c\left(f_\sigma(h_{i_1} \otimes h_{i_2}) - h_{i_1} \otimes h_{i_2}\right), \text{ for } 1 \le i_1 < i_2 \le n,$$

$$c\left(g_{\sigma,*}(h_{i_1} \otimes h_{i_2}) - h_{i_1} \otimes h_{i_2}\right), \text{ for } (i_1, i_2) \in \{1, \cdots, n\}^2,$$

$$c\left(f_\sigma(h_{i_1} \otimes h_{i_2} \otimes h_{i_3}) - h_{i_1} \otimes h_{i_2} \otimes h_{i_3}\right), \text{ for } 1 \le i_1 < i_2 < i_3 \le n,$$

$$c\left(g_{\sigma,*}(h_{i_1} \otimes h_{i_2} \otimes h_{i_3}) - h_{i_1} \otimes h_{i_2} \otimes h_{i_3}\right), \text{ for } (i_1, i_2, i_3) \in \{1, \cdots, n\}^3,$$

where "$*$" runs through all non-empty subsets of $\{1,2,3\}$, and additionally the *triangle relations*:

$$x_{ijk}^2 = x_{ij}x_{ik}x_{jk} - x_{ij}^2 - x_{ik}^2 - x_{jk}^2 + 4 \quad (1 \le i < j < k \le n),$$

and the *hexagon relations*:

$$x_{123}x_{ijk} - \frac{1}{2}\det \begin{bmatrix} x_{1i} & x_{1j} & x_{1k} \\ x_{2i} & x_{2j} & x_{2k} \\ x_{3i} & x_{3j} & x_{3k} \end{bmatrix} = 0 \quad (1 \le i < j < k \le n).$$

Then let $\mathcal{SL}^{(2)}(\sigma)$ be the ideal of $\mathcal{C}_n^{(2)}$ generated by

$$c\left(f_\sigma(h_{i_1} \otimes h_{i_2}) - h_{i_1} \otimes h_{i_2}\right), \text{ for } 1 \le i_1 < i_2 \le n,$$

$$c\left(g_{\sigma,*}(h_{i_1} \otimes h_{i_2}) - h_{i_1} \otimes h_{i_2}\right), \text{ for } (i_1, i_2) \in \{1, \cdots, n\}^2.$$

Moreover let $\mathcal{SL}^{(1)}(\sigma) := \langle 0 \rangle$. Note that $\mathcal{SL}^{(d)}(\sigma)$ $(d = 1,2)$ can be considered as restrictions of $\mathcal{SL}^{(3)}(\sigma)$ to $\mathcal{C}_n^{(d)}$ $(d = 1,2)$. Now, we have the following three ideals $\mathcal{SL}^{(d)}(\sigma)$ $(d = 1,2,3)$:

- $\mathcal{SL}^{(3)}(\sigma) := \langle c(f_\sigma(h) - h),\ c(g_{\sigma,*}(h) - h),\ \text{triangle, hexagon}$
- $\mathcal{SL}^{(2)}(\sigma) := \mathcal{C}_n^{(2)} \cap \mathcal{SL}^{(3)}(\sigma) = \langle c(f_\sigma(h) - h),\ c(g_{\sigma,*}(h) - h) \rangle$
- $\mathcal{SL}^{(1)}(\sigma) := \mathcal{C}_n^{(1)} \cap \mathcal{SL}^{(3)}(\sigma) = \langle 0 \rangle$

Definition 2.3 (**algebraic variety** $\mathcal{F}^{(d)}(\sigma)$ $(d = 1,2,3)$)**.** *For* $d = 1,2,3$, *we define* $\mathcal{F}^{(d)}(\sigma)$ *by the common zeros of the ideal* $\mathcal{SL}^{(d)}(\sigma)$ *with multiplicity.*

Note that the first variety $\mathcal{F}^{(1)}(\sigma)$ is always $\mathbb{C}$ for any $\sigma \in \mathcal{B}_n$.

3. Fundamental properties of the varieties $\mathcal{F}^{(d)}(\sigma)$

Theorem 3.1 (invariance under the Markov moves). *For $d = 1, 2, 3$ and $\sigma \in \mathcal{B}_n$, the three varieties $\mathcal{F}^{(d)}(\sigma_n\sigma)$, $\mathcal{F}^{(d)}(\sigma)$ and $\mathcal{F}^{(d)}(\sigma_j^{-1}\sigma\sigma_j)$ are isomorphic as algebraic variety.*

Therefore $\mathcal{F}^{(d)}(K) := \mathcal{F}^{(d)}(\sigma)$ $(d = 1, 2, 3)$ are knot invariants (up to isomorphism/coordinate change). The first variety $\mathcal{F}^{(1)}(K)$ is a trivial invariant, because $\mathcal{F}^{(1)}(K) = \mathbb{C}$. Note that the third variety $\mathcal{F}^{(3)}(K)$ can be computed easily by using $\mathcal{F}^{(2)}(K)$ by definition. In particular, the number of generators of $\mathcal{SL}^{(d)}(\sigma)$ $(d = 2, 3)$ and the triangle and the hexagon relations show the following dimensional property:

Remark 3.2 (dimensional property). *For any knot $K \subset S^3$,*

$$\dim_{\mathbb{C}}\left(\mathcal{F}^{(2)}(K)\right) \geq \dim_{\mathbb{C}}\left(\mathcal{F}^{(3)}(K)\right) \geq 0.$$

Note that the number of irreducible components of $\mathcal{F}^{(d)}(K)$ does not depend on the choice of the coordinates of the varieties. The viewpoint gives knot invariants taking its value in the non-negative integers $\mathbb{Z}_{\geq 0}$, which is easy to handle. So we define the following notion:

Definition 3.3 (cardinality of varieties)**.** *For an algebraic variety V with multiplicity, we denote the number of irreducible components of V with multiplicity by $\mathcal{N}(V)$, called the cardinality of V.*

Proposition 3.4. *For any knot $K \subset S^3$, $\mathcal{F}^{(d)}(K^{-1})$, $\mathcal{F}^{(d)}(K^*)$ and $\mathcal{F}^{(d)}(K)$ are isomorphic as algebraic variety, where K^{-1} is K with the opposite orientation and K^* is the mirror image of K. Therefore the cardinality does not change under reversing orientation and taking the mirror image.*

We calculated the variety $\mathcal{F}^{(2)}(K)$ by using a program running on Maple V. For the data of the cardinality $\mathcal{N}(\mathcal{F}^{(2)}(K))$, please refer to [N3].

Now, in the case of a knot K with a braid presentation of two strings, we can determine all the variety $\mathcal{F}^{(d)}(K)$ $(d = 1, 2, 3)$.

Proposition 3.5 (variety $\mathcal{F}^{(2)}(\sigma_1^q)$ for $\sigma_1 \in \mathcal{B}_2$). *For a $(2, q)$-torus knot $T(2, q)$, $\dim_{\mathbb{C}}(\mathcal{F}^{(2)}(T(2, q))) = 0$ and $\mathcal{N}(\mathcal{F}^{(2)}(T(2, q))) - 1 = \frac{q-1}{2}$.*

More generally, we have the following properties of $\mathcal{F}^{(3)}(K)$.

Theorem 3.6 (main result 1). *For a knot K with $\dim(\mathcal{F}^{(2)}(K)) = 0$, we have $\mathcal{N}(\mathcal{F}^{(3)}(K)) - 1 \geq \frac{|\Delta_K(-1)|-1}{2}$. The equality holds for 2-bridge knots.*

Let $A_K(m, l)$ be the A-polynomial of a knot K (for more information, please refer to [CCGLS]).

Theorem 3.7 (main result 2). *If a small knot K satisfies $\dim_{\mathbb{C}}\left(\mathcal{F}^{(2)}(K)\right) = 0$ then we have*

$$\mathcal{N}(\mathcal{F}^{(3)}(K)) - 1 \geq \textit{max-deg}_l(A_K(\sqrt{-1}, l)) - \textit{min-deg}_l(A_K(\sqrt{-1}, l))$$

Moreover K has no meridional boundary slopes, then

$$\mathcal{N}(\mathcal{F}^{(3)}(K)) - 1 \geq \textit{max-deg}_l(A_K(m, l))$$

The inequality is the best possible for 2-bridge knots.

The above main theorems imply that the variety $\mathcal{F}^{(3)}(K)$ is closely related to the $SL(2,\mathbb{C})$-character variety of the knot group G_K (refer to Section 4). In the next section, we give a proof of the main theorems by showing one of the relationships of $\mathcal{F}^{(3)}(K)$ with the $SL(2,\mathbb{C})$-character variety.

4. Relationships of $\mathcal{F}^{(d)}(K)$ with the $SL(2,\mathbb{C})$-character variety

Let $X(G) := \mathrm{Hom}(G, SL(2,\mathbb{C}))/\sim_{\mathrm{trace}}$. This is called the $SL(2,\mathbb{C})$-character variety of G. Note that the equivalent relation $\sim_{\mathrm{trace}}$ means that two representations ρ_1 and ρ_2 are identified if $\mathrm{trace}\rho_1(g) = \mathrm{trace}\rho_2(g)$ for all the element $g \in G$. Let

$$\begin{aligned} S_0(K) &:= X(G_K) \cap_{\mathrm{mul}} \{\mathrm{trace}(\rho(\mu)) = 0\} \ (\mu : \text{meridian}) \\ &= \{[\rho] \in X(G_K) \mid \mathrm{trace}(\rho(\mu)) = 0\} \end{aligned}$$

Note that the notation $\cap_{\mathrm{mul}}$ means the intersection with multiplicity. The following proposition is the key to Theorems 3.6 and 3.7:

Proposition 4.1. *For any knot $K \subset S^3$, we have $\mathcal{F}^{(3)}(K) \supset S_0(K)$.*

By using Proposition 4.1, and the following proposition, we can immediately get Theorems 3.6 and 3.7.

Proposition 4.2 (properties of the section $S_0(K)$). *For a knot K satisfying $\dim_{\mathbb{C}}(S_0(K)) = 0$, $\mathcal{N}(S_0(K)) - 1 \geq \frac{|\Delta_K(-1)|-1}{2}$. Moreover K is small, then*

$$\mathcal{N}(S_0(K)) - 1 \geq \textit{max-deg}_l(A_K(\sqrt{-1}, l)) - \textit{min-deg}_l(A_K(\sqrt{-1}, l)).$$

Since Proposition 4.2 is a consequence of the result in [N2], we omit the proof (for more information, please refer to [N1, N2]).

5. Topological framework for $\mathcal{F}^{(d)}(K)$: proof of Proposition 4.1

In this section, we give a sketch of a proof of Proposition 4.1 by using the topological framework of the variety $\mathcal{F}^{(3)}(K)$. For more information, please refer to the paper [N1].

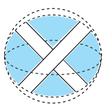
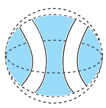
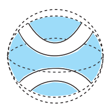
5.1. **Kauffman bracket skein module.** In fact, Proposition 4.1 can be immediately shown by using the topological framework of the variety $\mathcal{F}^{(3)}(K)$, i.e., *the Kauffman bracket skein module* (KBSM for short) [P]. Let $\mathbb{C}_t := \mathbb{C}[t, t^{-1}]$. For a compact orientable 3-manifold M, let

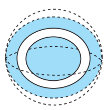
$$\mathcal{L}_t(M) := \mathrm{Span}_{\mathbb{C}_t}\{\text{all the isotopy classes of framed links } (\ni \phi) \text{ in } M\}$$

The KBSM $\mathcal{K}_t(M)$ of a 3-manifold M is defined as the quotient

$$\mathcal{K}_t(M) := \mathcal{L}_t(M)/\langle \textbf{Kauffman bracket skein relations at } t \rangle$$

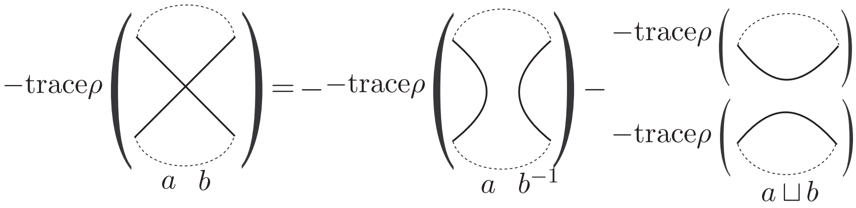
The Kauffman bracket skein relations at t are defined by the followings:

 $- t$  $- t^{-1}$ 

$L \sqcup$  $- (-t^2 - t^{-2})L$, for any framed link L in M

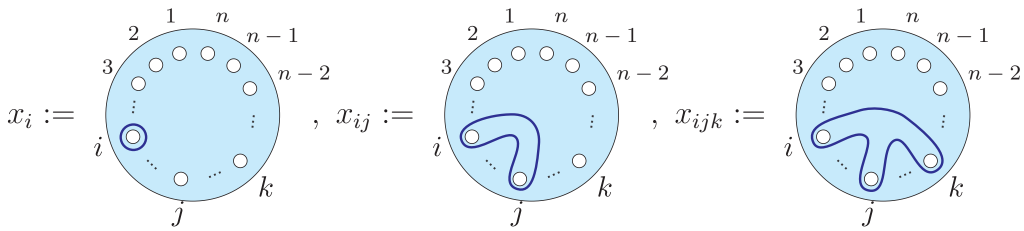
Note that at $t = -1$, the sign of a crossing and the framing of a link can be ignored.

5.2. **Visualization theorem (KBSM at $t = -1$).** The disjoint union gives a multiplicative operator in $\mathcal{K}_{-1}(M)$, thus $\mathcal{K}_{-1}(M)$ becomes a $\mathbb{C}$-algebra. In fact, the Kauffman bracket skein relation at $t = -1$ corresponds to *the $SL(2,\mathbb{C})$ Cayley-Hamilton identity*: $\mathrm{trace}\rho(ab) = -\mathrm{trace}\rho(ab^{-1}) + \mathrm{trace}\rho(a)\cdot\mathrm{trace}\rho(b)$. Namely, the Cayley-Hamilton identity can be visualized by using the Kauffman bracket skein relations at $t = -1$:

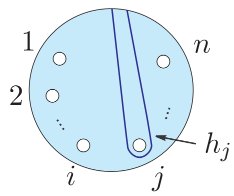


By using the visualization, we have the following consequence, which is a dual version of theorem in [B, PS]. As $\mathbb{C}$-module, the KBSM at $t = -1$ can be described by $\mathcal{K}_{-1}(M) = \mathcal{C}_N^{(3)}/\mathcal{I}(M)$, where $\mathcal{I}(M)$ is an ideal of $\mathcal{C}_N^{(3)}$. Then as a consequence of the results in [B, PS], we have $\mathrm{Sol}_{\mathbb{C}}(\mathcal{I}(M)) = X(\pi_1(M))$, where $\mathrm{Sol}_{\mathbb{C}}(\mathcal{I}(M))$ is the common zeros of the ideal $\mathcal{I}(M)$.

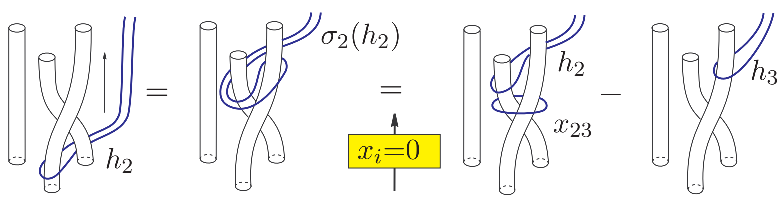
5.3. **Topological framework of the twisted homomorphisms f_σ and $g_{\sigma,*}$.** In fact, the twisted homomorphisms f_σ and $g_{\sigma,*}$ ($\sigma \in \mathcal{B}_n$) can be realized by an action of braids on the KBSM of a handlebody $\mathcal{H}_n$ of genus n at $t = -1$ with a condition called *trace-free condition*. Actually, the KBSM $\mathcal{K}_{-1}(\mathcal{H}_n = D_n^2 \times [0,1])$ is a $\mathbb{C}$-module generated by the following knots x_i, x_{ij} and x_{ijk} on the n-punctured disk $D_n^2 \times \{1\}$:

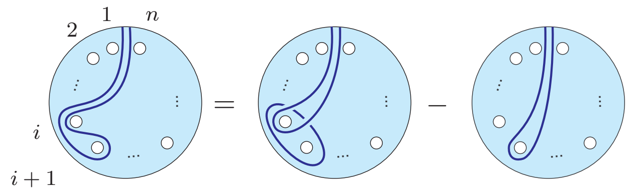


More precisely, $\mathcal{K}_{-1}(\mathcal{H}_n) = \mathcal{C}_n^{(3)}/\mathcal{I}_n$, where $\mathcal{I}_n$ is an ideal of $\mathcal{C}_n^{(3)}$. Now, we decompose the generators $x_{i_1\cdots i_k}$ into the following simple curves h_i on $D_n^2 \times \{1\}$:



Note that the projection $c : \mathcal{A}_n^{(3)} \to \mathcal{C}_n^{(3)}$ is the recovering operation of $x_{i_1\cdots i_k}$ from h_i's. Then it turns out that if we add the relations $x_i = 0$ $(1 \le i \le n)$, called the trace-free condition, to the KBSM $\mathcal{K}_{-1}(\mathcal{H}_n)$, we have





$$\sigma_i(h_i) = x_{ii+1}h_i - h_{i+1}$$
$$\sigma_i(h_j) = h_{p_{ii+1}(j)} \ (j \neq i)$$

In fact, we can show that for any knot K with a braid presentation $\sigma \in \mathcal{B}_n$ the KBSM $\mathcal{K}_{-1}(E_K) = \mathcal{C}_n^{(3)}/\mathcal{I}(E_K)$ of the knot exterior E_K can be presented by the quotient of $\mathcal{K}_{-1}(\mathcal{H}_n) = \mathcal{C}_n^{(3)}/\mathcal{I}_n$, and thus $\mathcal{I}(E_K) \supset \mathcal{I}_n$. Since we are now assuming the trace-free condition $x_i = 0$, $\mathcal{I}(E_K)_{\text{meridian}=0} \supset \mathcal{I}_{n,x_i=0}$ follows[3]. Here, we can show by using the above observation that $f_\sigma(h) - h$ and $g_{\sigma,*}(h) - h$ $(h \in \mathcal{A}_n^{(3)})$ are elements in the ideal $\mathcal{I}(E_K)_{\text{meridian}=0}$ via the projection $c : \mathcal{A}_n^{(3)} \to \mathcal{C}_n^{(3)}$. Moreover the triangle and the hexagon relations turn out to be contained in the ideal $\mathcal{I}_{n,x_i=0} \subset \mathcal{I}(E_K)_{\text{meridian}=0}$. Hence we have $\mathcal{SL}^{(3)}(\sigma) \subset \mathcal{I}(E_K)_{\text{meridian}=0}$, and thus

$$\mathcal{F}^{(3)}(K) = \mathrm{Sol}_{\mathbb{C}}^{\mathrm{mul}}(\mathcal{SL}^{(3)}(\sigma)) \supset \mathrm{Sol}_{\mathbb{C}}^{\mathrm{mul}}(\mathcal{I}(E_K)_{\text{meridian}=0}) = S_0(K),$$

where $\mathrm{Sol}_{\mathbb{C}}^{\mathrm{mul}}(A)$ means the common zeros of A with multiplicity. This shows Proposition 4.1.

6. Knot contact homology and the second ideal

The most surprising and unexpected property is that the second ideal can be thought of as the degree 0 abelian knot contact homology introduced by L. Ng via the differential graded algebra in [Ng1, Ng2]. I would like to thank Professor X.-S. Lin for letting me know the papers.

6.1. **Differential graded algebra and knot contact homology.** In [Ng1], Ng constructed a graded algebra (DGA) by using the Magnus representation of braids. (For the geometric background of DGA, refer to Section 3 of [Ng1].)

Let $\mathcal{A}_n$ be the tensor algebra on $n(n-1)$ generators a_{ij} $(1 \leq i, j \leq n,\ i \neq j)$. (Note that there is no relationships between $\mathcal{A}_n$ and $\mathcal{A}_n^{(d)}$ defined in Section 2.) Then we denote the group of automorphisms of $\mathcal{A}_n$ by $\mathrm{Aut}(\mathcal{A}_n)$.

For each generator σ_k of $\mathcal{B}_n$, define the homomorphism $\phi_{\sigma_k} \in \mathrm{Aut}(\mathcal{A}_n)$ by

$$\phi_{\sigma_k} := \begin{cases} a_{ki} & \mapsto & -a_{k+1i} - a_{k+1k} \otimes a_{ki}, & i \neq k, k+1 \\ a_{ik} & \mapsto & -a_{ik+1} - a_{ik} \otimes a_{kk+1}, & i \neq k, k+1 \\ a_{k+1i} & \mapsto & a_{ki}, & i \neq k, k+1 \\ a_{ik+1} & \mapsto & a_{ik}, & i \neq k, k+1 \\ a_{kk+1} & \mapsto & a_{k+1k} & \\ a_{k+1k} & \mapsto & a_{kk+1} & \\ a_{ij} & \mapsto & a_{ij}, & i, j \neq k, k+1 \end{cases}$$

In fact, the map $\phi : \mathcal{B}_n \to \mathrm{Aut}(\mathcal{A}_n)$ defined by $\phi(\sigma_k) := \phi_{\sigma_k}$ turns out to be a homomorphism. We consider the inclusion $i : \mathcal{B}_n \to \mathcal{B}_{n+1}$. This can be considered as just adding $(n+1)$-st strand which does not intersect with the other n strands. Then a_{in+1} and a_{n+1i} are denoted by a_{i*} and a_{*i}, respectively.

[3]Here $\mathcal{I}_{y=0}$ means the ideal $\mathcal{I}$ with the condition $y = 0$.

Now we consider an extension of the automorphism ϕ,

$$\phi^{\mathrm{ext}} := \phi \circ i : \mathcal{B}_n \to \mathcal{B}_{n+1} \to \mathrm{Aut}(\mathcal{A}_n).$$

By the definition of ϕ_σ, we can show that for a braid $\sigma \in \mathcal{B}_n$ and an indeterminate a_{i*} (resp. a_{*i}), $\phi_\sigma^{\mathrm{ext}}(a_{i*})$ (resp. $\phi_\sigma^{\mathrm{ext}}(a_{*i})$) is a linear combinations of a_{j*} (resp. a_{*j}).

Definition 6.1. *For a braid $\sigma \in \mathcal{B}_n$, the $(n \times n)$-matrices $\Phi_\sigma^L(A)$ and $\Phi_\sigma^R(A)$ are defined by*

$$\phi_\sigma^{\mathrm{ext}}(a_{i*}) = \sum_{j=1}^{n} (\Phi_\sigma^L(A))_{ij} a_{j*}, \ \phi_\sigma^{\mathrm{ext}}(a_{*i}) = \sum_{j=1}^{n} (\Phi_\sigma^R(A))_{ij} a_{*j}.$$

Definition 6.2 (degree 0 knot contact homology (Proposition 4.2 in [Ng1]))**.** *For a knot K in 3-sphere S^3, the degree 0 contact homology of K, written $HC_0(K)$, is the quotient algebra $\mathcal{A}_n/\mathcal{I}_\sigma$, where σ is a braid presentation of K, $\mathcal{I}_\sigma$ is the two-sided ideal of $\mathcal{A}_n$ generated by the entries of two matrices $(\Phi_\sigma^L(A) - E) \cdot A$ and $A \cdot (\Phi_\sigma^R(A) - E)$, $A := (a_{ij})$ and $E := \mathrm{diag}(1, \cdots, 1)$.*

Then abelianize the above framework by taking the quotient by $a_{ij} - a_{ij}$ and $a_{ij} \otimes a_{kl} - a_{kl} \otimes a_{ij}$ (i.e., the tensor becomes the usual commutative multiplication and $\mathcal{A}_n$ becomes a polynomial ring over $\mathbb{Z}$ or $\mathbb{C}$). We denote the abelianized algebra by $\mathcal{A}_n^{\mathrm{ab}}$.

Definition 6.3 (degree 0 abelian knot contact homology)**.** *For a knot K in real Euclidean space $\mathbb{R}^3$ (or 3-sphere S^3), the degree 0 abelian contact homology, written $CH_0^{\mathrm{ab}}(K)$, is defined by*

$$CH_0^{\mathrm{ab}}(K) := \mathcal{A}_n^{\mathrm{ab}}/\mathcal{I}_\sigma^{\mathrm{ab}},$$

where σ is a braid presentation of K, $\mathcal{I}_\sigma^{\mathrm{ab}}$ is the ideal of $\mathcal{A}_n$ generated by the entries of the two matrices $(\Phi_\sigma^{\mathrm{ab},L}(A) - E) \cdot A$ and $A \cdot (\Phi_\sigma^{\mathrm{ab},R}(A) - E)$, $A := (a_{ij})$ and $E := \mathrm{diag}(1, \cdots, 1)$.

Note that the matirx $\Phi_\sigma^{\mathrm{ab},R}(A)$ is in fact the transpose of $\Phi_\sigma^{\mathrm{ab},L}(A)$. It turns out that the degree 0 (abelian) contact homology of K does not depend on the choice of braid presentations of K (see Theorem 4.10 in [Ng1]). So $CH_0(K)$ and CH_0^{ab} are well-defined and thus invariants of knots.

6.2. **Efficient description of the ideal $\mathcal{SL}^{(2)}(\sigma)$.** We recall the ideal $\mathcal{SL}^{(2)}(\sigma)$ defined in Section 2. For a braid presentation $\sigma \in \mathcal{B}_{n\geq 2}$ of a knot K, $\mathcal{SL}^{(2)}(\sigma)$ is the ideal of $\mathcal{C}_n^{(2)}$ generated by

$$f_\sigma(x_{i_1 i_2}) - x_{i_1 i_2}, \ \text{for } 1 \leq i_1 < i_2 \leq n,$$

$$c \circ g_{\sigma,*}(h_{i_1} \otimes h_{i_2}) - x_{i_1 i_2}, \ \text{for } (i_1, i_2) \in \{1, \cdots, n\}^2.$$

Actually, it turns out that

$$f_\sigma(x_{i_1 i_2}) - x_{i_1 i_2}$$

for $1 \leq i_1 < i_2 \leq n$ can be described by

$$c \circ g_{\sigma,*}(h_{i_1} \otimes h_{i_2}) - x_{i_1 i_2}, \ \text{for } (i_1, i_2) \in \{1, \cdots, n\}^2.$$

Namely, the first type of generators can be omitted. To show this, we describe the automorphism $g_{\sigma,*}$ by the matrix presentation as the similar fashion to the 0-degree abelian knot contact homology. By definition, $f_\sigma(h_i)$ can be described by a linear combination of h_j's over $\mathcal{C}_n^{(2)}$. Then we define two $(n \times n)$-matrices G_σ^L and G_σ^R in $M_n(\mathcal{C}_n^{(2)})$ by

$$g_{\sigma,k}(h_i \otimes h_k) = (\sigma(h_i) \otimes h_k) = \textstyle\sum_{j=1}^n (G_\sigma^L)_{ij} h_j \otimes h_k,$$
$$g_{\sigma,i}(h_i \otimes h_k) = (h_i \otimes \sigma(h_k)) = \textstyle\sum_{j=1}^n (G_\sigma^R)_{jk} h_i \otimes h_j.$$

Note that G_σ^R is the transpose of G_σ^L. We next consider the automorphism f_σ. By definition, we have

$$f_\sigma(x_{ij}) = c \circ f_\sigma(h_i \otimes h_j) = c(f_\sigma(h_i) \otimes f_\sigma(h_j)).$$

Here we focus on $f_\sigma(h_i) \otimes f_\sigma(h_j)$.

$$f_\sigma(h_i) \otimes f_\sigma(h_j) = \sum_{k=1}^n (G_\sigma^L)_{ik} h_k \otimes \sigma(h_j) = \sum_{k=1,l=1}^n (G_\sigma^L)_{ik} (G_\sigma^R)_{lj} h_k \otimes h_l.$$

Hence we have the following matrix presentation of $f_\sigma(x_{ij})$:

$$(f_\sigma(x_{ij})) = G_\sigma^L \cdot X \cdot G_\sigma^R,$$

where the matrix $X := (x_{ij})$. Then we can transform the matrix $(f_\sigma(x_{ij})) - E$ like this:

$$\begin{aligned}(f_\sigma(x_{ij})) - E &= G_\sigma^L \cdot X \cdot G_\sigma^R - E \\ &= (G_\sigma^L - E) \cdot X \cdot G_\sigma^R + X \cdot (G_\sigma^R - E).\end{aligned}$$

Note that the entries of the matrices $(G_\sigma^L - E) \cdot X \cdot G_\sigma^R$ and $X \cdot (G_\sigma^R - E)$ are in the ideal generated by the entries of the matrices $(G_\sigma^L - E) \cdot X$ and $X \cdot (G_\sigma^R - E)$. Therefore the ideal $\mathcal{SL}^{(2)}(\sigma)$ can be considered as the ideal generated by the entries of the two matrices $(G_\sigma^L - E) \cdot X$ and $X \cdot (G_\sigma^R - E)$, where $X := (x_{ij})$ and $E := \mathrm{diag}(1, \cdots, 1)$.

6.3. **The ideals $\mathcal{SL}^{(2)}(\sigma)$ and $\mathcal{I}_\sigma^{\mathrm{ab}}$.** We take a closer look at the description of the ideal $\mathcal{I}_\sigma^{\mathrm{ab}}$ via the $(n \times n)$-matrices $\Phi_\sigma^{\mathrm{ab},L}(A)$ and $\Phi_\sigma^{\mathrm{ab},R}(A)$ for $\mathcal{A}_n^{\mathrm{ab}}$:

$$\phi_\sigma^{\mathrm{ext}}(a_{i*}) = \sum_{j=1}^n (\Phi_\sigma^{\mathrm{ab},L}(A))_{ij} a_{j*}, \ \phi_\sigma^{\mathrm{ext}}(a_{*i}) = \sum_{j=1}^n (\Phi_\sigma^{\mathrm{ab},R}(A))_{ji} a_{*j}.$$

Then by definition, $CH_0^{\mathrm{ab}}(K) = \mathcal{A}_n^{\mathrm{ab}}/\mathcal{I}_\sigma^{\mathrm{ab}}$, where σ is a braid presentation of K.

Here is a little digression. Recall the ideal $\mathcal{I}_\sigma^{\mathrm{ab}}$ is generated by the entries of the two matrices $\Phi_\sigma^{\mathrm{ab},L}(A) \cdot A$ and $A \cdot \Phi_\sigma^{\mathrm{ab},R}(A)$, where $A = (a_{ij})$. Here note that we made the matrices $\Phi_\sigma^{\mathrm{ab},L}(A)$ and $\Phi_\sigma^{\mathrm{ab},R}(A)$ by using the linear description of the action of the braid σ to a_{i*} and a_{*j}. However, the matrix which $\Phi_\sigma^{\mathrm{ab},L}(A)$ and $\Phi_\sigma^{\mathrm{ab},R}(A)$ are multiplied by is $A = (a_{ij})$. This operation can be thought of as the projection $c : \mathcal{A}_n^{(2)} \to \mathcal{C}_n^{(2)}$.

Now we can consider a map

$$T : \mathcal{A}_n^{(2)}/\mathcal{SL}^{(2)}(\sigma) \to \mathcal{A}_n^{\mathrm{ab}}/\mathcal{I}_\sigma^{\mathrm{ab}},$$

defined by $T(x_{ij}) := -a_{ij}$, $T(1) := 1$. The map T turns out to be a homomorphism as $\mathbb{C}$-algebra. Moreover T is clearly a bijection. Therefore T is an isomorphism. Then we have the following conclusion:

Theorem 6.4 (main result 3). *For any knot K in 3-sphere S^3, the quotient polynomial ring $\mathcal{C}_n^{(2)}/\mathcal{SL}^{(2)}(\sigma)$ is isomorphic to the degree 0 abelian knot contact homology $CH_0^{\mathrm{ab}}(K)$, where σ is a braid presentation of K.*

7. Remarks

The main results on the varieties $\mathcal{F}^{(d)}(K)$ stated in this report seem to be a piece of the properties which $\mathcal{F}^{(d)}(K)$ has. It is very interesting to look into the relationship of the variety $\mathcal{F}^{(3)}(K)$ with the Casson-Lin invariant/knot signature [L] which is a *model (but not exactly)* of the variety. It is also interesting to know whether or not the variety $\mathcal{F}^{(3)}(K)$ coincides with the section $S_0(K)$ for any knots (refer to Proposition 4.1). This viewpoint will give an answer to the question: is the variety $\mathcal{F}^{(3)}(K)$ a *real* combinatorial realization of the section $S_0(K)$?

Acknowledgements

I would like to thank Xiao-Song Lin for very fruitful discussions, comments and encouragement as well as for his hospitality during my stay at the University of California, Riverside, fall 2004–spring 2006. I am also grateful to the organizers of the conference *Topology of knots IX* for giving me an opportunity to talk.

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