RECURSIVE FORMULA OF THE COLORED JONES POLYNOMIAL FOR THE FIGURE-EIGHT KNOT WITH 0-FRAMING

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ABSTRACT. In this paper for the figure-eight knot K with 0-framing, we derive the the recursive formula of the colored Jones polynomial, denoted by $J_n(K)$, with respect to the dimension n of $sl(2,\mathbb{C})$-representation space by using a method introduced by Gelca [Ge] and the result shown by Gelca and Sain [GS]. However Gelca's formula is "reducible" and so is our result above. In the final section, therefore, we apply a modified version shown in [N], which is more "essential" and "simpler" than the original, and get "simpler" recursive formula of the colored Jones polynomial of the figure-eight knot.

1. INTRODUCTION

As well known, the colored Jones polynomial of a framed link L in S^3, denoted by $J_n(L)$, is a quantum $(sl(2,\mathbb{C}),\rho)$-invariant of L, where n means the dimension of the representation ρ of $sl(2,\mathbb{C})$. That values in the Laurent polynomial ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. (Please refer to [L] in detail.)

Formulating the value of $J_n(L)$ for any framed link L is not easy work but the value of some knots has done, for example the trefoil knot, figure-eight knot and the Hopf link. (Please refer to [H, L].) In [H], the colored Jones polynomial of the figure-eight knot was formulated as follows: let K be the figure-eight knot, then

$$J_n(K) = [n]\sum_{i=0}^{\infty} q^{-in}(1-q^{n-1})(1-q^{n-2})\cdots(1-q^{n-i})(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+i}),$$

where $[n] = (q^{\frac{n}{2}} - q^{-\frac{n}{2}})/(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$. Note that the sum above has compact support for any non-negative integer n.

In [Ge], the recursive formula of the colored Jones polynomial of the trefoil knot was formulated by using the Kauffman bracket skein module (abbreviated to KBSM for short). In this paper, we will be also concentrated on formulating the recursive formula with respect to the figure-eight knot in the same way as [Ge]. However the formula given by Gelca in [Ge] is "reducible", and so is our result appear in Section 3. So we tried modifying Gelca's formula in [N] to make it more "essential" and "simpler" one and will apply the modified version in Section 4.

Date: First version: 10 January, 2003, Second: 8 April, 2003 Third: 3 August, 2003.
2000 *Mathematics Subject Classification.* Primary 57M27; Secondary 57M25.
Key words and phrases. Colored Jones polynomial, Kauffman bracket skein module.
The author has been supported by JSPS Research Fellowships for Young Scientists.

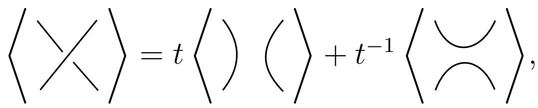
2. Colored Kauffman bracket

Now, the colored Jones polyonmial can be considered as the colored Kauffman bracket, which is also a non-oriented framed link invariant, by modification of the variable. (This will be mentioned below.) In this paper we will use the colored Kauffman bracket.

The n-th colored Kauffman bracket $\kappa_n(K)$ of a knot in S^3 is defined by the Kauffman bracket $\langle S_n(K)\rangle$ of the link $S_n(K)$ associated with the n-th Chebyshev polynomial $S_n(K)$ of K as follows:

$$S_{n+2}(K) = K \cdot S_{n+1}(K) - S_n(K), \quad S_1(K) = K, \quad S_0(K) = 1 \cdot \phi,$$

whereK^n for a framed knot K and any non-negative integer n, means the n-parallel of K. As well-known, the Kauffman bracket is a non-oriented framed link invariant valued in $\mathbb{C}[t, t^{-1}]$ defined by the skein relations as follows:



$$\langle L \sqcup \bigcirc \rangle = (-t^2 - t^{-2})\langle L \rangle,$$

where each link is identically embedded in S^3 except in $\mathrm{Int}(B^3)$ depicted above. In fact, for any framed knot K and any non-negative integer n, $\kappa_n(K)(t = -\sqrt{-1}q^{\frac{1}{4}})$ is equal to $J_{n+1}(K)$. Here extend $\kappa_n(K)$ to any integer by using the following relation: for any non-negative integer n,

$$\kappa_{-n}(K) = -\kappa_{n-2}(K).$$

This is a natural extension of $\kappa_n(K)$ in the sense of the recursion relation of the Chebyshev polynomial.

3. Recursive formula of the colored Jones polynomial for the figure-eight knot with 0-framing

3.1. **Main theorem and proof.** Now, let us derive the recursive formula of the colored Kauffman bracket of the figure-eight knot by using Gelca's formula in [Ge] and the result shown by Gelca and Sain in [GS].

Let K be the figure-eight knot with **0-framing** In this section, we denote $\langle K\rangle_n = \langle K\rangle_n(t)$ by $\kappa_n = \kappa_n(t)$ for convenience. Then the following theorem holds:

Proposition 3.1. *The recursive formula of the colored Kauffman brackets of the figure-eight knot $\{\kappa_m\}_m$ is*

$$a_n\kappa_{n+2} + b_n\kappa_{n+1} + c_n\kappa_n + d_n\kappa_{n-1} + e_n\kappa_{n-2} + f_n\kappa_{n-3} + g_n\kappa_{n-4} = 0,$$

with the following initial conditions:

$$\begin{aligned}
\kappa_{-4} &= -t^{-28} + t^{-20} - t^{-4} - 1 - t^4 + t^{20} - t^{28}\\
\kappa_{-3} &= t^{-10} + t^{10}\\
\kappa_{-2} &= -1\\
\kappa_{-1} &= 0\\
\kappa_0 &= 1\\
\kappa_1 &= -t^{-10} - t^{10}\\
\kappa_2 &= t^{-28} - t^{-20} + t^{-4} + 1 + t^4 - t^{20} + t^{28}.
\end{aligned}$$

Here a_n through g_n are as follows:

$$
\begin{aligned}
a_n &= -t^{6n+6} + t^{-2n+2} \\
b_n &= -t^{14n+24} + t^{10n+16} + t^{6n+20} - t^{6n+12} + t^{6n+8} + t^{6n+4} - t^{2n+12} + t^{2n+8} + t^{2n-4} \\
&\quad +t^{-2n+8} - 2t^{-2n+4} - t^{-2n} + t^{-2n-4} - t^{-2n-12} - t^{-6n+4} - t^{-6n-8} + t^{-10n-16}, \\
c_n &= -t^{14n+22} + 2t^{10n+18} - t^{10n+14} + t^{10n+6} + t^{6n+18} - t^{6n+14} + t^{6n+10} + t^{6n+6} \\
&\quad -t^{6n+2} + t^{6n-6} - t^{2n+14} + t^{2n+10} - t^{2n+2} - t^{2n-2} - t^{-2n+10} - t^{-2n-2} \\
&\quad -t^{-2n-6} + t^{-6n+6} - t^{-6n+2} + t^{-6n-2} + t^{-6n-6} - t^{-6n-10} + 2t^{-10n-2} - t^{-10n-6} \\
&\quad +t^{-10n-14} - t^{-14n-6}, \\
d_n &= t^{14n-4} + t^{10n+4} - t^{10n-4} - t^{6n+16} - t^{6n+8} - t^{6n+4} + t^{6n} - t^{6n-4} - t^{6n-8} + t^{2n+12} \\
&\quad -t^{2n+8} - 2t^{2n+4} + t^{2n} - 2t^{2n-8} - t^{-2n+12} + t^{-2n+8} + 2t^{-2n+4} - t^{-2n} + 2t^{-2n-8} \\
&\quad +t^{-6n+16} + t^{-6n+8} + t^{-6n+4} - t^{-6n} + t^{-6n-4} + t^{-6n-8} - t^{-10n+4} + t^{-10n-4} \\
&\quad -t^{-14n-4}, \\
e_n &= t^{14n-6} - 2t^{10n-2} + t^{10n-6} - t^{10n-14} - t^{6n+6} + t^{6n+2} - t^{6n-2} - t^{6n-6} + t^{6n-10} \\
&\quad +t^{2n+10} + t^{2n-2} + t^{2n-6} + t^{-2n+14} - t^{-2n+10} + t^{-2n+2} + t^{-2n-2} - t^{-6n+18} \\
&\quad +t^{-6n+14} - t^{-6n+10} - t^{-6n+6} + t^{-6n+2} - t^{-6n-6} - 2t^{-10n+18} + t^{-10n+14} - t^{-10n+6} \\
&\quad +t^{-14n+22}, \\
f_n &= -t^{10n-16} + t^{6n+4} + t^{6n-8} - t^{2n+8} + 2t^{2n+4} + t^{2n} - t^{2n-4} + t^{2n-12} + t^{-2n+12} \\
&\quad -t^{-2n+8} - t^{-2n-4} - t^{-6n+20} + t^{-6n+12} - t^{-6n+8} - t^{-6n+4} - t^{-10n+16} + t^{-14n+24}, \\
g_n &= -t^{2n+2} + t^{-6n+6}.
\end{aligned}
$$

Proof. First we review Gelca's formula for the recursion relation of the colored Kauffman bracket. In [Ge], Gelca proved the following formula: for an arbitrary knot K in S^3 an arbitrary non-zero element $\sum_{i=1}^{k} c_i(p_i, q_i)_T$ in $\mathrm{Ker}(\pi)$,

$$
\begin{aligned}
& \sum_{i=1}^{k} c_i t^{(2n+p_i)q_i}[(-t^2)^{q_i}\kappa_{n+p_i}(K) - (-t^{-2})^{q_i}\kappa_{n+p_i-2}(K)] \\
+ \; & \sum_{i=1}^{k} c_i t^{-(2n-p_i)q_i}[(-t^2)^{q_i}\kappa_{-n+p_i}(K) - (-t^{-2})^{q_i}\kappa_{-n+p_i-2}(K)] = 0.
\end{aligned}
$$

Note that $\mathcal{K}_t(T^2 \times I)$ has a basis $\{T_m((p,q)) = (mp, mq), 1 = \gcd(p,q)\}$ as $\mathbb{C}[t, t^{-1}]$-module, where (p,q) means the (p,q)-torus knot and $T_m((p,q))$ is the Chebyshev polynomial of (p,q) defined by

$$T_{m+2}((p,q)) = (p,q) \cdot T_{m+1}((p,q)) - T_m((p,q)),\ T_1((p,q)) = (p,q),\ T_0((p,q)) = 2.$$

Note that $(p,q)^a$ for any positive integer a is the a-parallel of (p,q). (Please refer to [Ge] in detail.)

In the case of the figure-eight knot, the kernel of π was determined by Gelca and Sain in [GS]. We pick up an element in $\mathrm{Ker}(\pi)$ below,

$$\begin{array}{rl} & t^{-6}(2,3)_T - t^{6}(2,-1)_T + t^{3}(1,7)_T - t(1,5)_T \\ + & (-t^{11}+t^{3}-t^{-1}-t^{-5})(1,3)_T + (t^{9}-t^{5}-t^{-7})(1,1)_T \\ + & (-t^{11}+2t^{7}+t^{3}-t^{-1}+t^{-9})(1,-1)_T + (t^{13}+t)(1,-3)_T \\ - & t^{-1}(1,-5)_T + t^{8}(0,7)_T + (-2t^{8}+t^{4}-t^{-4})(0,5)_T \\ + & (-t^{12}+t^{8}-t^{4}-1+t^{-4})(0,3)_T + (t^{12}-t^{8}+1+t^{-4})(0,1)_T. \end{array}$$

Applying this element in $\mathrm{Ker}(\pi)$ to Gelca's formula, we get the recursive formula of $\{\kappa_m\}_m$. (This calculation is complicated and so please pay attention to the calculation. I failed in this calculation no less than 3 times....) □

4. Modification of the formula of the figure-eight

As stated in the first section, our resulte in the previous section is "reducible". In this section, we will introduce the modified formula below. A proof of the formula will be given in another paper [N].

Proposition 4.1. *For an arbitrary element* $\sum_{i=1}^{k} c_i(p_i,q_i)_T \in \mathrm{Ker}(\pi_t)$,

$$\sum_{i=1}^{k} c_i t^{-p_iq_i}(-t^{2(n+p_i)+2})^{q_i}\kappa_{n+p_i} + \sum_{i=1}^{k} c_i t^{-p_iq_i}(-t^{2(n-p_i)+2})^{-q_i}\kappa_{n-p_i} = 0.$$

By using this formula, we get the recursion relation with length 4 easily. First pick up an element

$$\begin{array}{rl} & t^{-6}(2,3)_T - t^{6}(2,-1)_T + t^{3}(1,7)_T - t(1,5)_T + (-t^{11}+t^{3}-t^{-1}-t^{-5})(1,3)_T \\ + & (t^{9}-t^{5}-t^{-7})(1,1)_T + (-t^{11}+2t^{7}+t^{3}-t^{-1}+t^{-9})(1,-1)_T \\ + & (t^{13}+t)(1,-3)_T - t^{-1}(1,-5)_T + t^{8}(0,7)_T + (-2t^{8}+t^{4}-t^{-4})(0,5)_T \\ + & (-t^{12}+t^{8}-t^{4}-1+t^{-4})(0,3)_T + (t^{12}-t^{8}+1+t^{-4})(0,1)_T. \end{array}$$

Then this derives the recursion relation with length 5,

$$a_n\kappa_{n+2} + b_n\kappa_{n+1} + c_n\kappa_n + d_n\kappa_{n-1} + e_n\kappa_{n-2} = 0,$$

with the initial conditions $\kappa_{-2}=-1,\ \kappa_{-1}=0,\ \kappa_0=1$ and $\kappa_1=-t^{-10}-t^{10}$, where sequences a_n through e_n are as below:

$$\begin{array}{rcl} a_n &=& -t^{6n+6}+t^{-2n+2}, \\ b_n &=& -t^{14n+24}+t^{10n+16}+t^{6n+20}-t^{6n+12}+t^{6n+8}+t^{6n+4}-t^{2n+12}+t^{2n+8}+t^{2n-4} \\ && +t^{-2n+8}-2t^{-2n+4}-t^{-2n}+t^{-2n-4}-t^{-2n-12}-t^{-6n+4}-t^{-6n-8}+t^{-10n-16}, \\ c_n &=& -t^{14n+22}-t^{-14n-6}+2t^{10n+18}-t^{10n+14}+t^{10n+6}+2t^{-10n-2}-t^{-10n-6}+t^{-10n-14} \\ && +t^{6n+18}-t^{6n+14}+t^{6n+10}+t^{6n+6}-t^{6n+2}+t^{-6n+6}-t^{-6n+2}+t^{-6n-2}+t^{-6n-6} \\ && -t^{-6n-10}-t^{2n+14}+t^{2n+10}-t^{2n+2}-t^{2n-2}-t^{-2n+10}+t^{-2n+6}-t^{-2n-2}-t^{-2n-6}, \\ d_n &=& -t^{-14n-4}+t^{-10n-4}+t^{-6n+8}-t^{-6n}+t^{-6n-4}+t^{-6n-8}-t^{-2n+8}+t^{-2n+4} \\ && +t^{-2n-8}+t^{2n+12}-2t^{2n+8}-t^{2n+4}+t^{2n}-t^{2n-8}-t^{6n+16}-t^{6n+4}+t^{10n+4}, \\ e_n &=& -t^{-6n-6}+t^{2n+6}. \end{array}$$

Next, consider that the figure-eight knot is amphicheral and so we can pick up the element following element in $\mathrm{Ker}(\pi_t)$,

$$\begin{aligned} & t^6(2,-3)_T - t^{-6}(2,1)_T + t^{-3}(1,-7)_T - t^{-1}(1,-5)_T \\ + & (-t^{-11}+t^{-3}-t^1-t^5)(1,-3)_T + (t^{-9}-t^{-5}-t^7)(1,-1)_T \\ + & (-t^{-11}+2t^{-7}+t^{-3}-t^1+t^9)(1,1)_T + (t^{-13}+t^{-1})(1,3)_T - t^1(1,5)_T + t^{-8}(0,7)_T \\ + & (-2t^{-8}+t^{-4}-t^4)(0,5)_T + (-t^{-12}+t^{-8}-t^{-4}-1+t^4)(0,3)_T \\ + & (t^{-12}-t^{-8}+1+t^4)(0,1)_T. \end{aligned}$$

Then we get the following recursion relation with length 5,

$$a_n{}'\kappa_{n+2} + b_n{}'\kappa_{n+1} + c_n{}'\kappa_n + d_n{}'\kappa_{n-1} + e_n{}'\kappa_{n-2} = 0,$$

with the initial conditions $\kappa_{-2} = -1,\ \kappa_{-1} = 0,\ \kappa_0 = 1$ and $\kappa_1 = -t^{-10} - t^{10}$, where the sequences $a_n{}'$ through $e_n{}'$ are as below:

$$\begin{aligned} a_n{}' &= -t^{-6n-6} + t^{2n-2} \\ b_n{}' &= -t^{-14n-24} + t^{-10n-16} + t^{-6n-20} - t^{-6n-12} + t^{-6n-8} + t^{-6n-4} - t^{-2n-12} + t^{-2n-8} \\ &\quad +t^{-2n+4} + t^{2n-8} - 2t^{2n-4} - t^{2n} + t^{2n+4} - t^{2n+12} - t^{6n-4} - t^{6n+8} + t^{10n+16} \\ c_n{}' &= -t^{-14n-22} - t^{14n+6} + 2t^{-10n-18} - t^{-10n-14} + t^{-10n-6} + 2t^{10n+2} - t^{10n+6} + t^{10n+14} \\ &\quad +t^{-6n-18} - t^{-6n-14} + t^{-6n-10} + t^{-6n-6} - t^{-6n-2} + t^{6n-6} - t^{6n-2} + t^{6n+2} + t^{6n+6} \\ &\quad -t^{6n+10} - t^{-2n-14} + t^{-2n-10} - t^{-2n-2} - t^{-2n+2} - t^{2n-10} + t^{2n-6} - t^{2n+2} - t^{2n+6} \\ d_n{}' &= -t^{14n+4} + t^{10n+4} + t^{6n-8} - t^{6n} + t^{6n+4} + t^{6n+8} - t^{2n-8} + t^{2n-4} + t^{2n+8} + t^{-2n-12} \\ &\quad -2t^{-2n-8} - t^{-2n-4} + t^{-2n} - t^{-2n+8} - t^{-6n-16} - t^{-6n-4} + t^{-10n-4} \\ e_n{}' &= -t^{6n+6} + t^{-2n-6} \end{aligned}$$

Combining these two recursion relations, we get the following one with length 4,

$$\tilde{b_n}\kappa_{n+1} + \tilde{c_n}\kappa_n + \tilde{d_n}\kappa_{n-1} + \tilde{e_n}\kappa_{n-2} = 0,$$

with the initial conditions $\kappa_{-2} = -1,\ \kappa_{-1} = 0,\ \kappa_0 = 1$ and $\kappa_1 = -t^{-10} - t^{10}$, where $\tilde{b_n}$ through $\tilde{e_n}$ is as below,

$$\begin{aligned} \tilde{b_n} &= -t^{6n+4} - t^{10n+4} + t^{2n} + t^{6n} + t^{2n+8} - t^{-2n+4} + t^{-2n+8} - t^{-6n+4} \\ \tilde{c_n} &= -t^{2n+14} - t^{-14n-6} + t^{6n+6} - t^{2n+2} + t^{10n+10} - 2t^{-2n-6} - t^{-2n+10} + t^{-6n-2} \\ &\quad -2t^{6n+14} - t^{-10n-6} + 2t^{-10n-2} + t^{10n+18} + t^{-6n-10} - t^{14n+14} - t^{18n+14} + t^{10n+2} \\ &\quad +t^{-2n+2} - t^{2n+6} - t^{2n-6} + t^{-6n+6} + t^{6n+18} + t^{-2n-10} + 2t^{14n+10} - t^{6n-2} \\ \tilde{d_n} &= -t^{2n+16} + t^{2n-4} + t^{-6n+4} + t^{10n+4} + t^{10n+12} - t^{10n+8} + t^{14n+12} - t^{18n+12} - t^{2n} \\ &\quad -t^{6n} - t^{2n+8} - t^{-6n} + t^{10n} - t^{-2n+8} - t^{-14n-4} + t^{-10n-4} - 2t^{2n+4} + t^{10n+16} \\ &\quad +t^{-6n+8} + t^{-6n-4} + t^{-6n-8} + t^{2n+12} - t^{2n-8} \\ \tilde{e_n} &= -t^{-6n-6} + t^{2n+6} - t^{10n+14} + t^{2n+2} \end{aligned}$$

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