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THE MINIMAL RELATION IN THE KAUFFMAN BRACKET SKEIN MODULE OF THE m-TWIST KNOT

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ABSTRACT. In this paper, we will give an underlying relation in the Kauffman bracket skein module of the m-twist knot exterior, called "the minimal relation". Actually, this relation comes from a handle sliding of a knot in a handlebody of genus two along the 2-handle, when decomposing the m-twist knot exterior into a handlebody of genus two and a 2-handle. In the final section, we will give an application of the minimal relation to the $SL(2,\mathbb{C})$-character variety of the m-twist knot.

1. BACKGROUND AND MOTIVATION

In 1998, Frohman, Gelca and LoFaro introduced a non-commutative knot invariant, called the A-ideal, which is a left ideal of the non-commutative ring $\mathbb{C}_t[l,m]$ with the relation $lm = t^2ml$ ([FGL]). The A-ideal gives us another view point of the A-polynomial ([CCGLS]) and the colored Jones polynomial. We are now interested in "non-triviality" of the A-ideal. Here consider the map π_t

$$\pi_t : \mathcal{K}_t(T^2 \times I) \to \mathcal{K}_t(E_K)$$

induced by a "canonical" gluing of the cylinder over a torus to the knot exterior along their boundaries $T^2 \times \{1\}$ and ∂E_K. Actually, the A-ideal is associated with $\mathrm{Ker}(\pi_t)$, so we focus on $\mathrm{Ker}(\pi_t)$ instead of the A-ideal from now on. If $\mathrm{Ker}(\pi_t)$ of a knot K is non-trivial, that is, $\mathrm{Ker}(\pi_t)$ has a non-zero element, then we can get a recursion relation of the colored Jones polynomial $J_n(K)$ in terms of the dimension n of the $sl(2,\mathbb{C})$-representation ([G2]). This recursion relation has some information of the $SL(2,\mathbb{C})$-representations of the knot group. (Actually, we can derive information of the A-polynomial from this recursion relation.) Our interest is now headed toward the colored Jones polynomial given by a recursion relation coming from $\mathrm{Ker}(\pi_t)$. Let us first set the following notations:

(1) K: a knot in S^3,
(2) $E_K := S^3 - N(K)$, where $N(K)$ is an open tubular neighborhood of K,
(3) $R(\pi_1(E_K)) := \mathrm{Hom}_{\mathbb{C}}(\pi_1(E_K), SL(2,\mathbb{C}))$,
(4) $X(\pi_1(E_K)) := \{\text{character } \chi_\rho \text{ of } \rho \in R(\pi_1(E_K))\}$,
(5) $t_\gamma : X(\pi_1(E_K)) \to \mathbb{C},\ t_\gamma(\chi_\rho) := \mathrm{Trace}(\rho(\gamma))$, for $\gamma \in \pi_1(E_K)$,
(6) $\chi(\pi_1(E_K)) := \mathbb{C}[t_{\gamma_1}, t_{\gamma_2}, ..., t_{\gamma_k}]$, $(\gamma_i \in \pi_1(E_K))$,
(7) $\mathcal{K}_t(E_K)$: the Kauffman bracket skein module of E_K.

Here $R(\pi_1(E_K))$ and $X(\pi_1(E_K))$ in (3), (4) are called the $SL(2,\mathbb{C})$-representation variety and the $SL(2,\mathbb{C})$-character variety of the knot group $\pi_1(E_K)$ respectively. (Please refer to [CS] in detail.) $\chi(\pi_1(E_K))$ in (6) can be considered as the dual

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space of $X(\pi_1(E_K))$, and k depends on the number of generators of the knot group $\pi_1(E_K)$.

The Kauffman bracket skein module (KBSM for short) in (7) is defined in the beginning of the next section. Now under the above setting, roughly speaking, there exist the following relationships.

$$\begin{array}{ccc} \boxed{\mathcal{K}_{-1}(E_K)} = \boxed{\chi(\pi_1(E_K))} & \longleftrightarrow & \boxed{X(\pi_1(E_K))} \\ \uparrow \text{ projection } (t=-1) & & \uparrow \\ \boxed{\mathcal{K}_t(E_K)} = \boxed{\text{“noncommutative } \chi(\pi_1(E_K))\text{”}} & \longleftrightarrow & \boxed{\text{“virtual space”}} \end{array}$$

(Please refer to [B1] in detail.) Namely, we can regard the Kauffman bracket skein module $\mathcal{K}_t(E_K)$ as a “noncommutative $\chi(\pi_1(E_K))$”. Then, roughly speaking, the recursion relation of the colored Jones polynomial coming from $\mathrm{Ker}(\pi_t)$ represents a defining polynomial of a “noncommutative” $SL(2,\mathbb{C})$-character variety, which includes information of the A-polynomial of K. Therefore, if $\mathrm{Ker}(\pi_t)$ is non-trivial for any knot, then the colored Jones polynomial always represents a “noncommutative” $SL(2,\mathbb{C})$-character variety. It is uncertain whether or not $\mathrm{Ker}(\pi_t) \neq 0$ for any knot has not been shown yet. Hence we are interested in the “non-triviality” of $\mathrm{Ker}(\pi_t)$ and so the A-ideal.

On the other hand, Garoufalidis and Le recently showed the following theorem:

Theorem 1 (Garoufalidis-Le [GL])**.** *For any knot K, the colored Jones polynomial $J_n(K)$ has a recursion relation in terms of n.*

As well-known, the colored Jones polynomial is the quantum $sl(2,\mathbb{C})$-invariant. They showed this theorem by using some properties of the quantum group $\mathcal{U}_q(sl(2,\mathbb{C}))$ and some method in complex analysis. This is why we cannot derive information of the recursion relations coming from the Lie group $SL(2,\mathbb{C})$ from Garoufalidis-Le's theorem so far.

Now, in the case of the $(2,2p+1)$-torus knot, for any $p \in \mathbb{Z}$, the non-triviality of $\mathrm{Ker}(\pi_t)$ was shown ([GS]). To investigate the non-triviality of $\mathrm{Ker}(\pi_t)$ in other cases, we first focus on the case of the m-twist knot. Let K_m be the m-twist knot $(m \geq 0)$ as in Figure 1. (Here it suffices to observe the case where $m \geq 0$, since we can reduced the case where $m < 0$ to the above case by using the reflection.)

$K_m =$

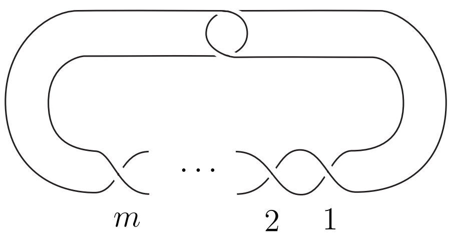


FIGURE 1. The m-twist knot K_m: the twists are ordered from the right to the left

Remark 1. *We see immediately that the diagram in Figure* 1 *is alternating and irreducible, so the minimal crossing number of K_m is exactly $m+2$. (Please refer to* [K, M, T] *in detail.)*

Recently the following result is announced in [BL].

Theorem 2 (Bullock-LoFaro [BL]).

$$\mathcal{K}_t(E_{K_m}) = \mathrm{Span}_{\mathbb{C}[t,t^{-1}]}\{x^i \cdot y^j | i \in \mathbb{Z}_{\geq 0}, m \geq j \geq 0\},$$

where x and y are knots in Figure 3 and the notation x^i (resp. y^j) means the i-parallel of x (resp. j-parallel of y) and $x^i \cdot y^j$ means x^i together with y^j.

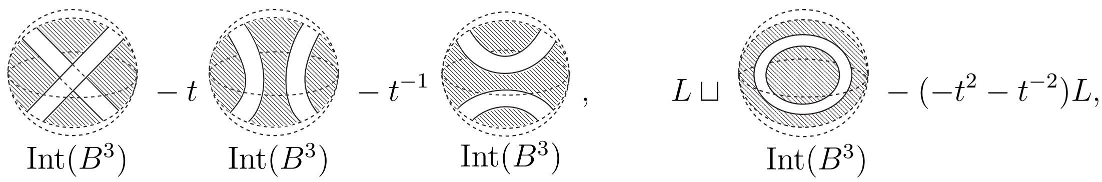
Remark 2. *The notation "$\cdot$" in the above theorem does not mean a "multiplication" in the KBSM $\mathcal{K}_t(E_{K_m})$. In general, a KBSM does not have multiplicative operations. However the notation makes sense as stated in Theorem 2.*

We cannot see the underlying diagrammatic structure of $\mathcal{K}_t(E_{K_m})$ in [BL], although the KBSM is a diagrammatic object. For example, we cannot see directly why y^{m+1} does not appear in $\mathcal{K}_t(E_{K_m})$.

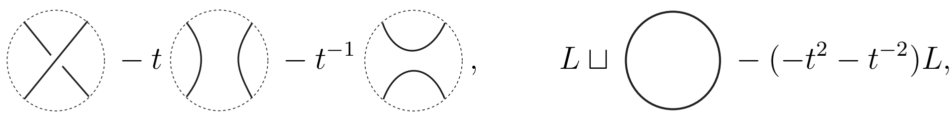
Our target in this paper is the relation in $\mathcal{K}_t(E_{K_m})$ which makes y^{m+1} into a sum of terms with lower degree in terms of y. (See Subsection 2.5.) This relation may help us to find a "simple" element of $\mathrm{Ker}(\pi_t)$ in the case of the m-twist knot, if $\mathrm{Ker}(\pi_t)$ is non-trivial. In the final section, we will show an application of the minimal relation to the $SL(2,\mathbb{C})$-character variety of K_m.

2. Minimal relation in the KBSM

2.1. **KBSM of 3-manifold.** For a compact orientable 3-manifold M, the Kauffman bracket skein module $\mathcal{K}_t(M)$ (KBSM for short) is defined. Namely, $\mathcal{K}_t(M)$ is the quotient of the $\mathbb{C}[t,t^{-1}]$-module $\mathbb{C}[t,t^{-1}]\mathcal{L}_M$ generated by all isotopy classes of framed links in M (including the empty link ϕ) by the $\mathbb{C}[t,t^{-1}]$-submodule generated by all possible the elements as follows:



where each depiction above is a diagram on an open disk  in $\mathrm{Int}(B^3)$ and each framed link in the first relation is identically embedded in M except in $\mathrm{Int}(B^3)$. The above relations introduced in $\mathbb{C}[t,t^{-1}]\mathcal{L}_M$ is called the Kauffman bracket skein relations. In this paper, we will treat only a knot exterior and a handlebody of genus two as M. In these cases, the framing is presented by the blackboard framing. Then the Kauffman bracket skein relations are simply depicted as follows:



An element of $\mathcal{K}_t(M)$ is called a skein. There exist some observed KBSM's, for instance, $F \times [0,1]$ where F is an orientable surface ([P]), twisted I-bundle of a non-orientable surface ([P]), the exterior of a knot in S^3 ([B2, BL]), lens spaces ([HP]) and so on.

Remark 3. *For any compact orientable 3-manifold M, the KBSM $\mathcal{K}_{-1}(M)$ at $t = -1$ has a "multiplicative operation" defined as a disjoint union. This operation is well-defined, since the sign of a crossing in a link can be ignored by the Kauffman bracket skein relation at $t = -1$. So $\mathcal{K}_{-1}(M)$ becomes an algebra.*

2.2. **KBSM of m-twist knot exterior.** Let K_m be the m-twist knot ($m \geq 0$) in S^3, and let H_2 be a handlebody of genus two. Since the tunnel number of K_m is 1, the exterior E_{K_m} has the following decomposition:

$$E_{K_m} = H_2 \cup (\text{2-handle}).$$

Here the following two theorems are the keys to investigate $\mathcal{K}_t(E_{K_m})$.

Theorem 3 (Przytycki [P]). *Under the same notations as in Theorem 2, the following holds:*

$$\mathcal{K}_t(H_2) = \mathrm{Span}_{\mathbb{C}[t,t^{-1}]}\{x^i y^j z^k \mid i,j,k \in \mathbb{Z}_{\geq 0}\},$$

where skeins x, y and z are as in Figure 2.

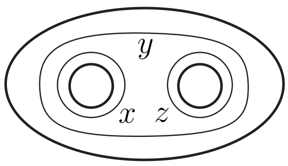


FIGURE 2. Skeins x, y and z in $\mathcal{K}_t(H_2)$

Theorem 4 (Przytycki [P]).

$$\mathcal{K}_t(E_K) = \mathcal{K}_t(H_2)/J,$$

where J is the submodule of $\mathcal{K}_t(H_2)$ generated by

$$\{L - sl(L) | L : \text{any framed link in } H_2\}.$$

The writing "sl" means the resulting link L after an arbitrary handle slide.

We see immediately that in $\mathcal{K}(E_{K_m})$ the skein z is identified with the skein x as seen in Figure 3 (via a handle slide relation).

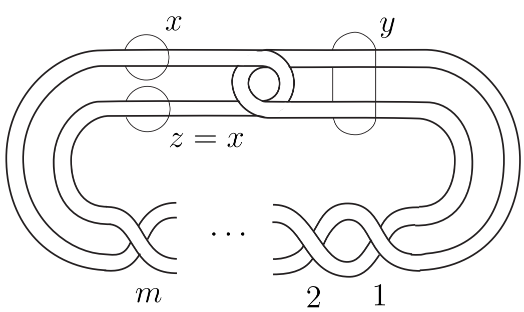
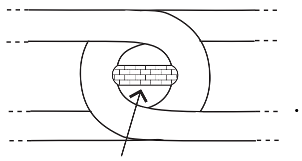


FIGURE 3. Skeins x, y and z in $\mathcal{K}_t(E_{K_m})$

2.3. **Attaching slope of $H_2 \cup$ (2-handle).** We will depict a attaching slope of $H_2 \cup$(2-handle). Let us drill a tunnel in the knot complement near the clasping part as below:



Then focus on the attaching slope of removed 2-handle and the resulting handlebody of genus two. A boundary of $S^1 \times I \subset D^2 \times I$ is attached to the thin curve depicted in Figure 4.

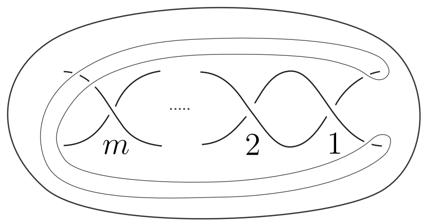


FIGURE 4. Attaching slope of $H_2 \cup$(2-handle): thick lines mean holes corresponding genera of H_2. The thin curve means the attaching slope and it lies on the boundary of H_2.

For convenience, we omit the outer thick curve corresponding to the boundary of H_2 from now. Regarding the attaching slope as a skein in $\mathcal{K}_t(H_2)$, we can consider any handle slide relation as a band sum of two skeins in $\mathcal{K}_t(H_2)$ along some band b.

2.4. **Minimal relation in the KBSM of twist knot.** First, we define the minimal relation in the KBSM of the m-twist knot.

Definition 1. *The minimal relation in $\mathcal{K}_t(E_{K_m})$ is the relation $R_m(t) = 0$ in $\mathcal{K}_t(E_{K_m})$ such that $R_m(t)$ is monic as an element of $\mathbb{C}[t, t^{-1}, x][y]$ and has minimal degree in terms of y in all the relations.*

The minimal relation is uniquely determined up to the powers of t and t^{-1}. By Theorem 2, the degree of $R_m(t)$ in terms of y is clearly $m+1$. The minimal relation $R_m(t) = 0$ does not generate all underlying relations in the KBSM $\mathcal{K}_t(E_{K_m})$. For example,

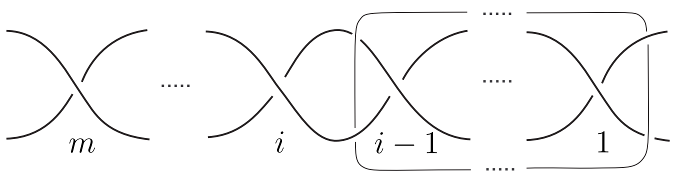
$$y^{m+2} = y \cdot y^{m+1} \neq y \cdot (y^{m+1} - R_m(t)).$$

However the relation at $t = -1$ means the defining polynomial of the KBSM at $t = -1$. Namely,

$$\mathcal{K}_{-1}(E_{K_m}) = \mathbb{C}[x, y]/\langle R_m(-1)\rangle,$$

not as module but as algebra, where $\langle R_m(-1)\rangle$ is the ideal in $\mathbb{C}[x, y]$ generated by $R_m(-1)$. This is shown by using the fact that $\mathcal{K}_t(E_{K_m})$ is free as module.

2.5. **Diagrammatic approach to the minimal relation.** We will discuss the skein X_i in $\mathcal{K}_t(H_2)$ defined as follows:

$X_i :=$ 

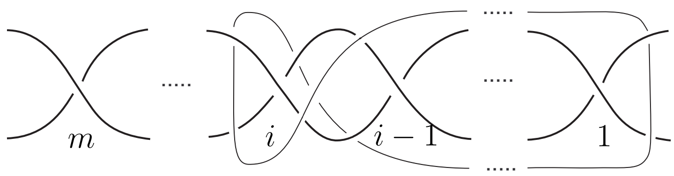
.

$$X_0 := (-t^2 - t^{-2})\phi.$$

Lemma 1 (Recursion relation of X_i)**.** *The skein X_i, $(m+1 \geq i \geq 0)$, as an element in $\mathcal{K}_t(E_{K_m})$ satisfies the following recursion relation:*

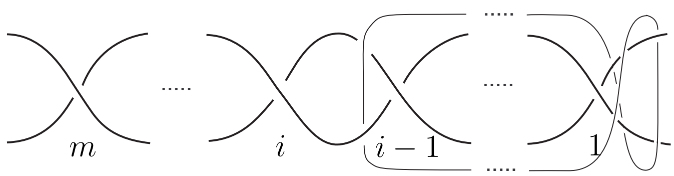
$$X_{i+2} - t^2 y X_{i+1} + t^4 X_i + 2t^2 x^2 = 0,\ X_1 = -t^2x^2 - t^4 y,\ X_0 = -t^2 - t^{-2}.$$

Proof. All of the operations below are done in the handlebody H_2. First, transform X_{i+1} as follows:

$(-t^{-3})$ 

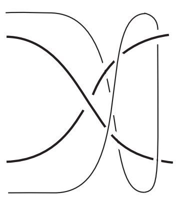
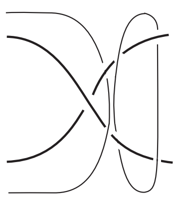
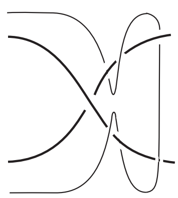
.

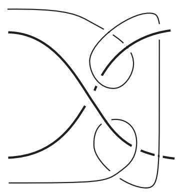
Then slide the kink to the right side as below:

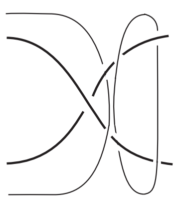
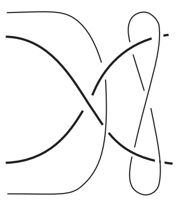
$(-t^{-3})$ 

.

Here resolve the crossing by using the skein relation. For example,

 $= t$  $+ t^{-1}$  ,

 $= (-t^3)^2$  ,

 $= (-t^3)$  .

Substituting $z = x$, we can get the recursion relation stated in Lemma 1. $\square$

Lemma 2 (General term)**.** *The skein X_i, $(m+1 \geq i \geq 0)$, as an element in $\mathcal{K}_t(E_{K_m})$ is formulated as follows:*

$$X_i = -t^{2(i-2)}(t^6 S_i(y) + t^4 S_{i-1}(y)x^2 - t^2 S_{i-2}(y)) - 2t^2x^2 \sum_{n=0}^{i-2} t^{2n} S_n(y),$$

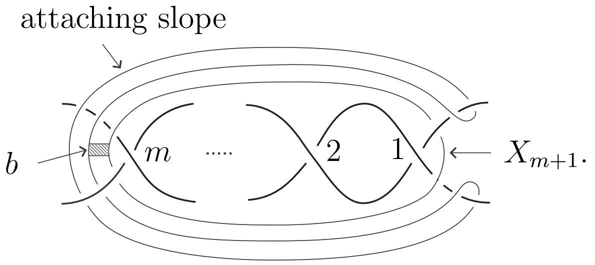
where $S_i(y)$ is the element in $\mathbb{C}[t, t^{-1}]\mathcal{L}_{H_2}$ defined recursively as follows:

$$S_{i+2}(y) = yS_{i+1}(y) - S_i(y),\ S_1(y) = y,\ S_0(y) = 1 \cdot \phi.$$

Lemma 3 (handle slide relations)**.** *For any non-negative integer m, X_m as an element in $\mathcal{K}_t(E_{K_m})$ has the following relation:*

$$X_{m+1} + t^{-4}X_m + t^{-2}x^2 = 0.$$

Proof. All of the operations below are done in the handlebody H_2. First, consider the following band sum of X_{m+1} and the attaching slope along a band b:



Let $sl_b(X_{m+1})$ be the resulting knot after the band sum. Then a relation $sl_b(X_{m+1}) - X_{m+1} = 0$ holds in $\mathcal{K}_t(E_{K_m})$. Resolving $sl_b(X_{m+1})$ and substituting $z = x$, we get,

$$sl_b(X_{m+1}) = -t^{-4}X_m - t^{-2}x^2.$$

This completes the proof. $\square$

Theorem 5. *For any non-negative integer m, the following relation is derived from the handle slide relation in Lemma 3:*

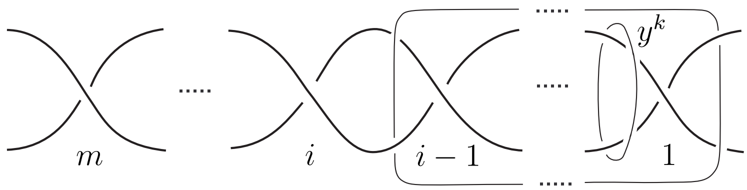
$$\begin{aligned} R_m(t) &:= S_{m+1}(y) + t^{-6}S_m(y) - t^{-4}S_{m-1}(y) - t^{-10}S_{m-2}(y) \\ &+ \left\{t^{-2}S_m(y) + (2t^{-4} + t^{-8})S_{m-1}(y)\right\} x^2 \\ &+ \left\{2(t^{-2m-2} + t^{-2m-6}) \sum_{i=0}^{m-2} t^{2i} S_i(y)x^2 - t^{-2m-6}\right\} x^2 \\ &= 0. \end{aligned}$$

Remark 4. $\deg_y(S_i(y)) = i$ *and $S_i(y)$ is monic.*

Recalling Theorem 2, we find that $R_m(t) = 0$ is the minimal relation in $\mathcal{K}_t(E_{K_m})$.

2.6. **How to get all the relations in terms of y.** In addition, we can get all the relations in terms of y. We will show the method below.

Let us focus on the following skein $y^k * X_i$, $(k \in \mathbb{Z}_{\geq 0},\ m+1 \geq i \geq 0)$, in $\mathcal{K}_t(H_2)$

$y^k * X_i :=$ 

, $(m+1 \geq i \geq 1)$,

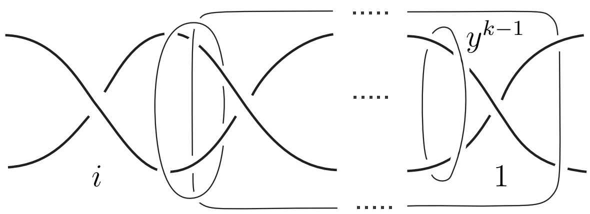
$$y^k * X_0 := y^k X_0.$$

Then we can show the following equation.

Lemma 4. *For $k \in \mathbb{Z}_{\geq 0}$, $i \in \{1, \cdots, m+1\}$, $y^k * X_i$ as an element of $\mathcal{K}_t(E_{K_m})$ satisfies the following relation:*

$$y^k * X_i = y^{k-1} * (t^4 y X_i + (-t^6 + t^2) X_{i-1} + 2(-t^4 + 1)x^2).$$

Proof. First, consider the following configuration of $y^k * X_i$.



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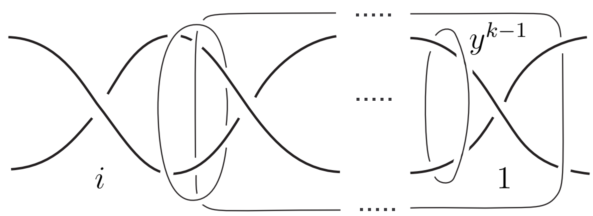
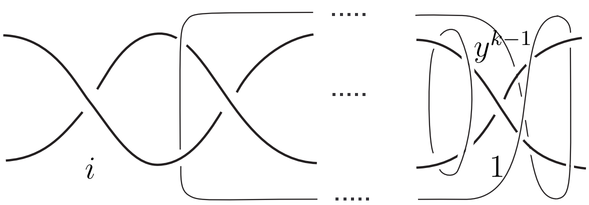
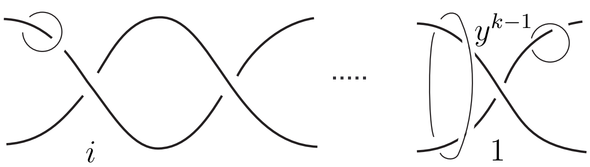
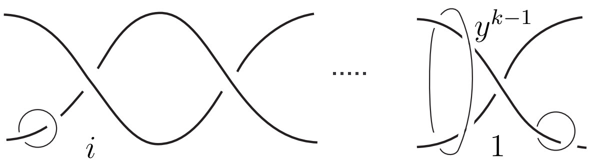
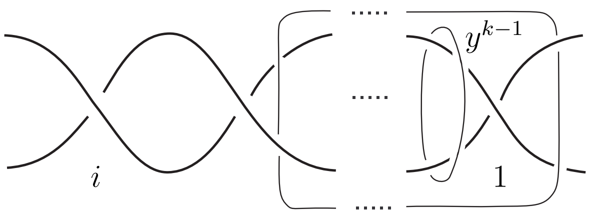
Then we can calculate $y^k * X_i$ as follows:

(1) $y^k * X_i \quad =$ 

(2) $= \quad -t^{-1}$ 

(3) $+$ 

(4) $+$ 

(5) $+\ t^{-2}$  .

The last term (5) in the above calculation is exactly $t^{-2}y^{k-1} * X_{i-1}$. Considering $x = z$, we see that (3) and (4) are x^2y^{k-1}. (2) can be calculated via resolution on page 46 and is

$$y^{k-1} * (t^4yX_i - t^6X_{i-1} - 2t^4x^2).$$

(Note that $y^k * x^j = x^jy^k$.) This completes the proof. $\square$

We can derive the relation in terms of y from the relation in Lemma 4 as follows. For any $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} y^k * X_{m+1} &= y^{k-1} * (t^4yX_{m+1} + (-t^6+t^2)X_m + 2(-t^4+1)x^2) \\ &= y^{k-2} * (t^4(y * X_{m+1}) \cdot y + (-t^6+t^2)y * X_m + 2(-t^4+1)x^2y) \\ &\cdots \\ &= \sum_{j=0}^{m+1} f_j(t,t^{-1},y)X_j, \end{aligned}$$

where $f_{m+1}(t,t^{-1},y) = t^{4k}y^k$. Here there exists no interaction between $f_j(t,t^{-1},y)$ and X_j in $f_j(t,t^{-1},y)X_j$, hence we can get its general term by the same way as Lemma 2. Namely, its general temrs is presented by $f_j(t,t^{-1},y)$ times the general term of X_j. We can check that a handle slide relation also holds for $y^k * X_{m+1}$ and is

$$y^k * X_{m+1} + t^{-4}y^k * X_m + t^{-2}x^2y^k = 0.$$

In the end, combining the above handle slide relation and the general term of $f_j(t,t^{-1},y)X_j$, we get the relation

$$y^{k+m+1} = \sum_{j=0}^{k+m} g_j(t,t^{-1},x)y^j.$$

This is how to calculate all the relations in terms of y.

3. Application to the character variety of the m-twist knot

We focus on Bullock's theorem.

Theorem 6 (Bullock [B1])**.** *For any compact orientable 3-manifold M, there exists a surjective homomorphism Φ as algebra*

$$\Phi : \mathcal{K}_{-1}(M) \to \chi(\pi_1(M)),$$

defined by $\Phi(K) := -t_{[K]}$, $\Phi(K_1 \sqcup \cdots \sqcup K_i) := \prod_{j=1}^{i} \Phi(K_i)$, where $[K]$ is an element of $\pi_1(M)$ represented by the knot K with an unspecified orientation. Moreover the kernel of Φ is the nilradical $\sqrt{0}$.

According to Bullock's theorem, $R_m(-1)$ has information of the defining polynomial of the character variety $X(\pi_1(E_{K_m}))$. By Maple, we can observe the following factorizations of $R_m(-1)$ over $\mathbb{Q}$:

$$\begin{aligned}
R_0(-1) &= y+2,\\
R_1(-1) &= (y+2)(y+x^2-1),\\
R_2(-1) &= (y+2)(y^2+x^2y-y+x^2-1),\\
R_3(-1) &= (y+2)(y^3+x^2y^2-y^2-2y+x^2y+1),\\
R_4(-1) &= (y+2)(y^4+x^2y^3-y^3-3y^2+x^2y^2-x^2y+2y+1),\\
R_5(-1) &= (y+2)(y^5+x^2y^4-y^4-4y^3+x^2y^3-2x^2y^2+3y^2+3y-x^2y+x^2-1).
\end{aligned}$$

In fact, we can get the following decomposition in general case.

Lemma 5. *For any non-negative integer m, the minimal relation $R_m(-1)$ has the following decomposition:*

$$(y+2)\left(S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)\right).$$

Moreover, the factor $S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)$ is irreducible over $\mathbb{Q}$.

Proof. The first statement can be shown by using the properties of the Chebyshev polynomial S_m. The second statement can be proved by a result on the trace field shown by J. Hoste and P. Shanahan ([HS]). Let us define the following notation:

$$\widetilde{R}_m(x,y) := S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y).$$

In the case of $m = 0, 1$, (that is the case of the unknot and the right-handed trefoil which are non-hyperbolic knots), it was observed above. Next, consider the case where $m \geq 2$. Then the twist knot K_m is hyperbolic. Hence there exists the discrete faithful representation

$$\rho_0 : \pi_1(E_{K_m}) \to SL(2,\mathbb{C})$$

of $\pi_1(E_{K_m})$. By Theorem 6, we can regard the skeins x and y as the functions $-t_x$ and $-t_y$, respectively. Here x is a meridional skein and the factor $y+2$ corresponds to the abelian representations of $\pi_1(E_{K_m})$, so we can assume that

$$x(\rho_0) = -t_x(\rho_0) = -2,\ y(\rho_0) = -t_y(\rho_0) = \alpha,$$

where α is a solution of $\widetilde{R}_m(-2, y) = 0$ over $\mathbb{C}$.

Now, by Corollary 1 in [HS] and Remark 1, the extension field $\mathbb{Q}(t_\gamma(\rho_0) : \gamma \in \pi_1(E_{K_m}))$ over $\mathbb{Q}$, called the trace field of K_m, has degree m. Namely,

$$[\mathbb{Q}(t_\gamma(\rho_0) : \gamma \in \pi_1(E_{K_m})) : \mathbb{Q}] = m.$$

Meanwhile, the degree of $\widetilde{R}_m(-2, y)$ in terms of y is m. Here we can easily show that $\mathbb{Q}(t_\gamma(\rho_0) : \gamma \in \pi_1(E_{K_m}))$ is simple extension, that is,

$$\mathbb{Q}(t_\gamma(\rho_0) : \gamma \in \pi_1(E_{K_m})) = \mathbb{Q}(\alpha).$$

(For example, show this using the fact $\chi(\pi_1(E_{K_m})) = \mathbb{C}[t_{[x]}, t_{[y]}]$, where $[x]$ and $[y]$ are elements in $\pi_1(E_{K_m})$ represented by the knots x and y, respectively.) Hence $\widetilde{R}_m(-2, y)$ must be irreducible over $\mathbb{Q}$. It is not so hard to see that if $\widetilde{R}_m(x, y)$ is reducible, then so is $\widetilde{R}_m(-2, y)$. These two facts complete the proof. □

By Lemma 5, we get the following result:

Theorem 7. *The KBSM $\mathcal{K}_{-1}(E_{K_m})$ has trivial nilradical. Therefore the following holds:*

$$\chi(\pi_1(E_{K_m})) = \mathbb{C}[x, y]/\langle R_m(-1)\rangle.$$

Hence the minimal relation $R_m(-1)$ at $t = -1$ is the defining polynomial of $X(\pi_1(E_{K_m}))$.

Note that by using Lemma 5 we can show the minimality of $R_m(t)$ without Theorem 2.

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