ON A TOPOLOGICAL ASPECT OF THE CHEBYSHEV POLYNOMIALS AND THE CHARACTER VARIETIES

FUMIKAZU NAGASATO

Dedicated to Professor Akio Kawauchi for his 60th birthday

Abstract. We show a topological aspect of the Chebyshev polynomials by discussing factorization of a polynomial over complex number field. Then it turns out that the polynomial of which we consider the factorization is the defining polynomial of the $SL_2(\mathbb{C})$-character variety of a twist knot group.

1. Introduction

In this paper, we focus on a family of polynomials $\{S_n(z)\}_{n \geq 0}$ in an indeterminate $z$ defined by the following recursive relation:

$$S_{n+2}(z) = zS_{n+1}(z) - S_n(z), \quad S_1(z) = z, \quad S_0(z) = 1.$$

The polynomial $S_n(z)$ can be transformed into the Chebyshev polynomial of the second type $U_n(z)$ defined by

$$U_{n+2}(z) = 2zU_{n+1}(z) - U_n(z), \quad U_1(z) = 2z, \quad U_0(z) = 1.$$

Indeed, $U_n(z) = S_n(2z)$ holds for any $n \geq 0$.

On the other hand, the family of polynomials $\{S_n(z)\}_{n \geq 0}$ can be derived from the Cayley-Hamilton identity for $SL_2(\mathbb{C})$ as follows. For elements $A$ and $B$ in $SL_2(\mathbb{C})$, by the Cayley-Hamilton identity we have

$$\text{tr}(AB) = \text{tr}(A)\text{tr}(B) - \text{tr}(AB^{-1}),$$

where “tr” means the trace of a matrix. Applying this identity to a matrix $A^2 \in SL_2(\mathbb{C})$, we obtain $\text{tr}(A^2) = \text{tr}(A)^2 - 2$. Similarly, $\text{tr}(A^{n+2}) = \text{tr}(A)\text{tr}(A^{n+1}) - \text{tr}(A^n)$ holds for any $n \geq 0$. Setting $T_n(\text{tr}(A)) := \text{tr}(A^n)$ and $z := \text{tr}(A)$, we obtain

$$T_{n+2}(z) = zT_{n+1}(z) - T_n(z), \quad T_1(z) = z, \quad T_0(z) = 2.$$

2000 Mathematics Subject Classification. Primary 57M27; Secondary 57M25.

Key words and phrases. character variety, Chebyshev polynomial, Kauffman bracket skein module.
Then the polynomial $S_n(z)$ for $n \geq 0$ is given by

\[
\begin{align*}
S_{2m}(z) &= \sum_{i=1}^{m} T_{2i}(z) + 1, & \text{if } n \text{ is even}, \\
S_{2m+1}(z) &= \sum_{i=1}^{m} T_{2i-1}(z), & \text{if } n \text{ is odd}.
\end{align*}
\]

Hence the polynomial $S_n(z)$ can be thought of as a polynomial in $\text{tr}(A)$ for an arbitrary $A \in \text{SL}_2(\mathbb{C})$.

In this sense, it is natural to use the family of polynomials $\{S_n(z)\}_{n \geq 0}$ or $\{T_n(z)\}_{n \geq 0}$ in the study of $\text{SL}_2(\mathbb{C})$-representations of a group. In the present paper, we will apply the polynomials $\{S_n(z)\}_{n \geq 0}$ to study the set of characters of $\text{SL}_2(\mathbb{C})$-representations of a knot group $G_K$ of a knot $K$, which is the fundamental group of the exterior $E_K$ of an open tubular neighborhood of $K$ in $3$-sphere. (A knot means the image of a circle under an embedding map into $3$-sphere. For more information, refer to [15] for example.)

The set of characters of $\text{SL}_2(\mathbb{C})$-representations of a finitely generated and presented group $G$ has a one-to-one correspondence to a closed algebraic set in some complex space. The algebraic set is denoted by $X(G)$ and called the $\text{SL}_2(\mathbb{C})$-character variety of a group $G$, sometimes simply called the character variety of $G$ (refer to Subsection 2.1 or [5]). Note that any knot group is finitely generated and presented. The character variety of a knot group has been playing important roles in low-dimensional topology. For example, the Culler-Shalen theory ([5]) is one of the most excellent applications of the character varieties. However the computation of the character variety is usually very hard. So it is interesting to find an alternative method to compute the character varieties.

Here we remark that the family of polynomials $\{S_n(z)\}$ is a key object to compute the character variety in the following topological way. In fact, the polynomial $S_n(z)$ had been discovered as the Jones-Wenzl idempotents [11] in low-dimensional topology. Roughly speaking, the idempotents can be thought of as topological operators to make coloring (certain parallel copies) of a simple closed curve on an annulus. This notion is based on the Kauffman bracket [10], which is an invariant of knots defined as a mathematical state model of the Jones polynomial [9]. Afterward, the Kauffman bracket of knots was generalized to an invariant of 3-manifolds, called the Kauffman bracket skein modules (KBSM for short). The KBSM $K_t(M)$ of a 3-manifold $M$ is a quotient of the $\mathbb{C}[t,t^{-1}]$-module generated by all the isotopy classes of embedded annuli in $M$ (refer to Subsection 2.2 or [13]). For example, in Theorem 2.1 of [3] Bullock and LoFaro found that the KBSM $K_t(E_{K_m})$ of the exterior $E_{K_m}$ for an $m$-twist knot $K_m$ (shown in Figure 1) can be described as $\mathbb{C}[t,t^{-1}]$-module by indeterminates $x$ and $y$ corresponding to the isotopy classes of annuli $\tilde{x}$ and $\tilde{y}$ shown in Figure 1:

\[
K_t(E_{K_m}) = \text{Span}_{\mathbb{C}[t,t^{-1}]} \{x^p y^q \mid p, q \in \mathbb{Z}_{\geq 0}, \ m \geq q\},
\]
where $x^m y^p$ means the isotopy class of the disjoint union of $m$ parallel copies of $\tilde{x}$ and $n$ parallel copies of $\tilde{y}$. In [8], we studied all the relations in the KBSM $\mathcal{K}_t(E_{K_m})$ by using the polynomials $\{S_n(z)\}_{n \geq 0}$. Then we found a relation $R_m(t) = 0$ in the KBSM $\mathcal{K}_t(E_{K_m})$ such that all the relations in $\mathcal{K}_t(E_{K_m})$ are given by using $R_m(t) = 0$:

**Theorem 1** (Theorem 4 in [8]). Let $R_m(t)$ be the following polynomial:

\[
S_{m+1}(y) + (t^6 - t^2x^2)S_m(y) + ((2t^4 + t^8)x^2 - t^{-4})S_{m-1}(y) \\
-t^{-10}S_{m-2}(y) + 2x^2(t^{-2m-2} + t^{-2m-6})\sum_{i=0}^{m-2} t^{2i}S_i(y) - t^{-2m-6}x^2.
\]

Then in $\mathcal{K}_t(E_{K_m})$, the polynomial satisfies $R_m(t) = 0$.

![m-twist knot](image)

**Figure 1.** $m$-twist knot $K_m$ in $S^3$ and annuli $\tilde{x}$ and $\tilde{y}$ in $E_{K_m}$

As shown in Theorem 1, the family of polynomials $\{S_n(z)\}_{n \geq 0}$ is also useful to describe algebraic structures of the KBSM $\{S_n(z)\}_{n \geq 0}$ simplifies the description of $R_m(t)$. We call the equation $R_m(t) = 0$ the minimal relation. In fact, the minimal relation $R_m(-1) = 0$ at $t = -1$ has the information of the defining polynomials of the character variety $X(G_{K_m})$ of the $m$-twist knot group $G_{K_m}$. Indeed, a specialized KBSM $\mathcal{K}_{-1}(E_{K_m})$ at $t = -1$ becomes a $\mathbb{C}$-algebra (see Subsection 2.2) and it follows from the result of [3] and [8] that $\mathcal{K}_{-1}(E_{K_m})$ is expressed by the following quotient polynomial ring:

\[
\mathcal{K}_{-1}(E_{K_m}) = \frac{\mathbb{C}[x, y]}{(R_m(-1))},
\]

where $(R_m(-1))$ means the principal ideal of $\mathbb{C}[x, y]$ generated by $R_m(-1)$. Then, considering Theorem 3 in Subsection 2.2 and the character ring $\chi(G_{K_m})$ of $G_{K_m}$, which is the coordinate ring of $X(G_{K_m})$ (refer to Subsection 2.1), we can obtain the following isomorphic correspondence as $\mathbb{C}$-algebra:

\[
\frac{\mathcal{K}_{-1}(E_{K_m})}{\sqrt{0}} \cong \chi(G_{K_m}).
\]
where the square root means the radical (thus $\sqrt{0}$ means the nilradical). 
Therefore the character ring $\chi(G_{K_m})$ is isomorphic to $\mathbb{C}[x, y]/\sqrt{(R_m(-1))}$. 
In the paper [8], we discussed the factorization of the polynomial $R_m(-1)$ 
to get the defining polynomial of $X(G_{K_m})$, which means the generator of 
$\sqrt{(R_m(-1))}$ uniquely determined up to non-zero constant multiple, by applying 
a property on radicals
\begin{equation}
\sqrt{\langle f_1^{a_1} \cdots f_n^{a_n} \rangle} = \langle f_1 \cdots f_n \rangle
\end{equation}
for polynomials $f_i$ in $\mathbb{C}[x, y]$ and non-negative integers $a_i$. Then we obtained 
the following:

**Proposition 1** (Lemma 7 in [8]). For any non-negative integer $m$, $R_m(-1)$ has the following decomposition:
\begin{equation}
(y + 2) \left( S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y) \right)
\end{equation}
Moreover, the factor $S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)$ is irreducible over the rational number field $\mathbb{Q}$.

To show Proposition 1 we used a certain topological information on $E_{K_m}$ (see [8]). Meanwhile, we had no methods to determine the factorization of $R_m(-1)$ over $\mathbb{C}$ at that time. Afterward, we found that the polynomial $R_m(-1)$ can be studied over $\mathbb{C}$ by using algebraic properties of the polynomial $S_n(z)$. The following is the main claim in this paper.

**Theorem 2.** Let $\tilde{R}_m(x, y)$ be $S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)$. Then $\tilde{R}_m(x, y)$ has no repeated factors in its factorization over $\mathbb{C}$. Hence
\begin{equation}
\chi(G_{K_m}) \cong \frac{\mathbb{C}[x, y]}{(R_m(-1))},
\end{equation}
and thus $R_m(-1)$ is the defining polynomial of $X(G_{K_m})$.

In the proof of Theorem 2, we show a topological aspect of the Chebyshev polynomials in topology.

2. Materials we need: character varieties and KBSMs

2.1. Character varieties. We quickly review some fundamental properties of the character varieties. For more information, refer to [5].

Let $G$ be a finitely generated and presented group. For a representation $\rho : G \to \text{SL}_2(\mathbb{C})$, the character $\chi_\rho$ of $\rho$ means a function on $G$ defined by $\chi_\rho(g) := \text{tr}(\rho(g))$, $g \in G$. If two representations $\rho_i : G \to \text{SL}_2(\mathbb{C})$ ($i = 1, 2$) are conjugate (i.e., there exists an element $A$ of $\text{SL}_2(\mathbb{C})$ such that $A^{-1} \rho_2(g) A = \rho_1(g)$ holds for any element $g$ in $G$), then $\chi_{\rho_1} = \chi_{\rho_2}$. Let $R(G)$ be the set of
representations $\rho: G \to \text{SL}_2(\mathbb{C})$. $R(G)$ is a non-empty set for any $G$ because there exists at least a trivial representation $\rho_0: G \to \text{SL}_2(\mathbb{C})$ defined by

$$\rho_0(g) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for any $g$ in $G$. For each element $g$ in $G$, we can define a function $t_g$ on $R(G)$ by $t_g(\rho) := \text{tr}(\rho(g)) = \chi_\rho(g)$. Let $T$ denote the ring generated by all the functions $t_g$, $g \in G$. By Proposition 1.4.1 in [5], the ring $T$ is finitely generated. So we can fix a finite set \{$g_1, \ldots, g_N$\} of $G$ such that $t_{g_1}, \ldots, t_{g_N}$ generate $T$. Consider a map

$$t: R(G) \to \mathbb{C}^N, \quad t(\rho) := (t_{g_1}(\rho), \ldots, t_{g_N}(\rho)), \quad \rho \in R(G).$$

Then the image $t(R(G))$ is denoted by $X(G)$ and called the $\text{SL}_2(\mathbb{C})$-character variety of $G$. Indeed, $X(G)$ is a closed algebraic set (refer to Corollary 1.4.5 in [5]). Since $t_{g_1}(\rho), \ldots, t_{g_N}(\rho)$ generate $T$, the character $\chi_\rho$ of a representation $\rho$ in $R(G)$ is determined by $t(\rho)$. Hence there exists a natural one-to-one correspondence between the points of $X(G)$ and the characters of representations in $R(G)$ (i.e., $X(G)$ can be identified with the set of characters of all $\rho$ in $R(G)$):

$$\{\chi_\rho \mid \rho \in R(G)\} \equiv X(G).$$

As $X(G)$ is an algebraic set, we can consider its coordinate ring as follows. Suppose $X(G)$ is an algebraic set in complex space $\mathbb{C}^m$ ($m > 0$). Let $I(X(G))$ be the ideal of the polynomial ring $\mathbb{C}[x_1, \ldots, x_m]$ that consists entirely of polynomials vanishing on $X(G)$. Then the coordinate ring of $X(G)$, denoted by $\chi(G)$, is defined as the quotient polynomial ring $\mathbb{C}[x_1, \ldots, x_m]/I(X(G))$, which is sometimes called the character ring of $G$. Note that the coordinate ring $\chi(G_{K_m})$ has trivial nilradical.

In the case of a knot group $G_K$, the coordinate ring $\chi(G_K)$ is isomorphic to a topological object given by a specialized KBSM reviewed in the next subsection.

2.2. KBSM from the $\text{SL}_2(\mathbb{C})$-character variety point of view. In this subsection, we give a short review on the Kauffman bracket skein module (KBSM), especially from the $\text{SL}_2(\mathbb{C})$-character variety point of view. For more information, refer to [13] and [14].

The KBSM $K_t(M)$ of a 3-manifold $M$ was introduced by Przyticki as a natural generalization of the Kauffman bracket to general 3-manifolds. More precisely, for a compact oriented 3-manifold $M$, let $L_t(M)$ be the $\mathbb{C}[t, t^{-1}]$-module generated by all the isotopy classes of embedded annuli in $M$ (including the empty set $\phi$). Then the KBSM $K_t(M)$ of the 3-manifold $M$ is defined
as the quotient of $\mathcal{L}_1(M)$ by the Kauffman bracket skein relations as below:

\[
L \sqcup - (t^2 - t^{-2})L,
\]

where $L$ is any embedded annuli in $M$ and “$\sqcup$” means the disjoint union. Note that the above depictions express annuli identically embedded in $M$ except in a ball illustrated with dashed curves. Each annulus is depicted with blackboard framing along a given disk in the ball. More precisely, suppose that an orientation is given on every annulus in $M$ and the disk in the ball. Then, in the above depictions, annuli must be placed on the disk so that the orientations of annuli coincide with that of the disk.

We now focus on a specialized KBSM $\mathcal{K}_{-1}(M)$ at $t = -1$, which is a key object in this paper. In $\mathcal{K}_{-1}(M)$, the following phenomena always happen:

\[
= - t \quad - t^{-1} \quad \quad = - (t^2 - t^{-2})L,
\]

So in $\mathcal{K}_{-1}(M)$ it suffices to consider homotopy types of annuli (moreover we can ignore the framing (twisting) of annuli by the second phenomenon). This gives us a guarantee that we can define a multiplication in $\mathcal{K}_{-1}(M)$ by the disjoint union $\sqcup$. Hence $\mathcal{K}_{-1}(M)$ becomes a $\mathbb{C}$-algebra and then the unit is the empty set $\phi$. This $\mathbb{C}$-algebra $\mathcal{K}_{-1}(M)$ was afterward linked to the character ring of the fundamental group $\pi_1(M)$ as follows:

**Theorem 3** (Bullock [2], Przytycki-Sikora [14]). For any compact orientable 3-manifold $M$, there exists a surjective homomorphism $\Phi$ as $\mathbb{C}$-algebra

\[
\Phi : \mathcal{K}_{-1}(M) \to \chi(\pi_1(M)),
\]

defined by $\Phi(K) := -t[K]$, $\Phi(K_1 \sqcup \cdots \sqcup K_i) := \prod_{j=1}^{i} \Phi(K_i)$, where $[K]$ is an element of $\pi_1(M)$ represented by the knot $K$ with an unspecified orientation and a base point. Moreover the kernel of $\Phi$ is the nilradical $\sqrt{0}$.

By Theorem 3, the polynomials \( \{S_n(z)\}_{n \geq 0} \) and the KBSM have been used to give an alternate description of the $\text{SL}_2(\mathbb{C})$-character varieties (refer to [8], [12] for example).
The alternative description using the KBSM has a nice property in the sense that it can naturally generalize the \( A \)-polynomial \([4]\), which is a knot invariant defined by using the character variety of a knot group, to a noncommutative setting (see \([7]\)). The computation of the noncommutative version of the \( A \)-polynomial relies heavily on good understandings of the KBSM of exterior of an open tubular neighborhood of a knot in 3-sphere. So Theorem 1 would be a powerful tool to compute the noncommutative \( A \)-polynomial.

3. Proof of Theorem 2: factorization of \( S_m(z) - S_{m-1}(z) \)

To prove Theorem 2 we do not use any topological properties of the twist knots but some algebraic properties of the polynomials \( \{S_n(z)\}_{n \geq 0} \).

**Proof.** It suffices to show that the polynomial \( \tilde{R}_m(x, y) \) has no repeated factors in its factorization over \( \mathbb{C} \) because we know the property (1) on radicals mentioned in Section 1.

Assume that \( \tilde{R}_m(x, y) \) has a repeated factor in the factorization over \( \mathbb{C} \). Then \( \tilde{R}_m(x, y) \) is reducible over \( \mathbb{C} \) (but irreducible over \( \mathbb{Q} \) by Proposition 1). In this situation, we have the following two cases for the factorization of \( \tilde{R}_m(x, y) \) in terms of the variable \( x \):

- \((ax + b)(cx + d)\), where \( a, b, c, d \in \mathbb{C}[y] \),
- \((ax^2 + b)c\), where \( a, b, c \in \mathbb{C}[y] \), \( ax^2 + b \) is irreducible over \( \mathbb{C} \).

We see that the first case never happens, because if it happens then the equation

\[
acx^2 + (ad + bc)x + bd = S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)
\]

must hold. So we have

\[
ac = \sum_{i=0}^{m-1} S_i(y), \quad ad + bc = 0, \quad bd = S_m(y) - S_{m-1}(y).
\]

Let us focus on the degree of the above three equations in terms of \( y \):

\[
\deg(a) + \deg(c) = m - 1, \quad \deg(a) + \deg(d) = \deg(b) + \deg(c),
\]

\[
\deg(b) + \deg(d) = m.
\]

Note that \( \deg(S_m(y)) \) is \( m \) by definition. Combining these equations, we have

\[
2 \deg(a) = 2 \deg(b) - 1,
\]

i.e., an even number equals an odd number, a contradiction.

We can also check that the second case never happens by using algebraic properties of the polynomials \( S_m(y) \) as follows. For the second case, the
equation
\[ ax^2 + bc = S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y). \]

must hold. In particular, the following equation is required:
\[ bc = S_m(y) - S_{m-1}(y). \]

Note that \( S_m(y) - S_{m-1}(y) \) has no repeated factors because the Chebyshev polynomial \( U_m(y) = S_m(2y) \) has the property \( U_m(\cos(\theta)) = \frac{\sin((m+1)\theta)}{\sin(\theta)} \) (refer to [1] for example). Namely, for any integer \( 0 \leq i \leq m - 1 \),
\[
S_m \left( 2 \cos \frac{2i + 1}{2m + 1} \pi \right) - S_{m-1} \left( 2 \cos \frac{2i + 1}{2m + 1} \pi \right) = \frac{1}{\sin \left( \frac{2i + 1}{2m + 1} \pi \right)} \left[ \sin \left( i + \frac{1}{2} + \frac{i + \frac{1}{2}}{2m + 1} \pi \right) - \sin \left( i + \frac{1}{2} - \frac{i + \frac{1}{2}}{2m + 1} \pi \right) \right] = 0
\]

Note that the degree of \( S_m(y) - S_{m-1}(y) \) in terms of \( y \) is \( m \) by definition. Thus we see that \( S_m(y) - S_{m-1}(y) \) has the following nice factorization:
\[ S_m(y) - S_{m-1}(y) = \prod_{i=0}^{m-1} \left( y - 2 \cos \left( \frac{2i + 1}{2m + 1} \pi \right) \right). \]

Hence \( bc \), especially the polynomial \( c \) has no repeated factors. Then the factor \( ax^2 + b \) must have a repeated factor, a contradiction. Therefore the polynomial \( \tilde{R}_m(x, y) \) has no repeated factors over \( \mathbb{C} \). Since it is clear that \( \tilde{R}_m(x, y) \) does not have the factor \( y + 2 \) (for example, use the fact that \( S_m(-2) - S_{m-1}(-2) \neq 0 \)), we have
\[ \sqrt{\langle \tilde{R}_m(-1) \rangle} = \langle R_m(-1) \rangle. \]

This completes the proof. \( \square \)

Acknowledgement

The author has been partially supported by Grant-in-Aid for Young Scientists (Start-up), Japan Society for the Promotion of Science.

References


Department of Mathematics, Meijo University, Tempaku, Nagoya 468-8502, Japan
E-mail address: fukky@ccmfs.meijo-u.ac.jp